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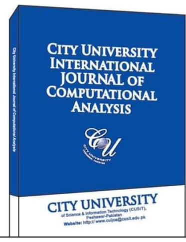

Efficient Quadrature Rules for Numerical Evaluation of Singular and Hyper-Singular Integrals

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ABSTRACT

In this study, hyper-singular and singular integrals with a singularity at the origin are numerically solved using quadratures based on hybrid functions, Haar wavelets, and the Newton-Cote quadrature rule. The Hadamard finite part (HFP) and Cauchy principle value (CPV) integrals are two examples of these integrals. To evaluate the effectiveness and precision of the novel approaches, the suggested regulations are numerically evaluated on a few problem instances.

1 Introduction

Several scientific and technical fields, such as fluid mechanics, aerodynamics, fracture mechanics, and wave propagation use singular integrals [4, 7, 8, 12]. Singular integrals can typically be expressed in the following way.

$$I_\gamma(g,x) = \int_a^b \frac{g(x)}{(x-\tau)^{\gamma+1}}dx, \quad a < \tau < b, \tag{1}$$

where,$g(x) \in [a,b]$. Owing to the pole of order$(\gamma+1)$ at$x = \tau \in [a,b]$, ordinary quadrature procedures such as the Simpson rule and Gaussian quadrature, among others, are unable to estimate the integral (**??**). For$\gamma = 0$, Eq. (1) is referred to as the Cauchy principle value (CPV) integral and can be written as:

$$I_0(g,x) = \int_a^b \frac{g(x)}{x-\tau}dx = \underset{\nu\to 0}{Lim}\left[\int_\nu^{\tau-\nu} \frac{g(x)}{x-\tau}dx + \int_{\tau+\nu}^b \frac{g(x)}{x-\tau}dx\right].$$

Equation (1) can be consistently written as and is known as the Hadamard finite part (HFP) integral for values of =1:

$$\begin{aligned} I_1(g,\tau) &= H\int_a^b \frac{g(x)}{(x-\tau)^2}dx = \underset{\nu\to 0^+}{Lim}\left[\int_\nu^{\tau-\nu} \frac{g(x)}{(x-\tau)^2}dx + \int_{\tau+\nu}^b \frac{g(x)}{(x-\tau)^2}dx\right] \\ &= H\int_a^b \frac{g(x)}{(x-\tau)^2}dx + \underset{\nu\to 0^+}{Lim}\frac{2g(x)}{\nu}. \end{aligned} \tag{2}$$

Many effective techniques for numerically evaluating the singular component have been devised [2, 5, 10, 11, 13, 20, 22]. Venturino [3] split the integration region into sub-domains and then used reduced order Gaussian type quadrature to analyse the HFP integral across each comment section. An approach for the numerical computation of HFP and CPV integrals has been created by Criscuolo [2]. For the numerical solution of CPV integrals, Pawel Keller employed the adaptable quadrature procedure [12].

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The researchers in reference [20] have substituted Lagrange's interpolating polynomial with Chebyshev-Gauss-libretto nodes for the phase function f(x) of the CPV integral. This approach separates the integral into non-singular and singular integrals. The singular portion is then computationally estimated. Subsequently, the investigators in reference [9] created a set of quadrature criteria of the Newton Cotes form for the numeric assessment of the HFP, and CPV.

The CPV integral problem was converted into a non-singular and an oscillatory singular integral in the article [6]. After performing the expansion of the Taylor series, the non-singular integral is assessed using an enhanced Levin approach, and the singular part is calculated mathematically. For the numerical computation of HFP integrals, a consistent estimate approach has lately been designed [23]. For simulating numerical results of triple, double, and single integral equations with fixed bounds, uniform Hybrid Functions quadrature, and Haar Wavelets procedures are established in the paper mentioned in reference [15]. The hybrid and Haar wavelets functions are expanded to assess triple, double, and single, integrals with arbitrary bounds in a subsequent study conducted by the same scholars [1]. Algorithms for the hybrid functions of order m=2,3,. . . 10 are inferred to properly assess the definite integrals with changeable constraints. For the numerical computation of one-dimensional immensely oscillating integrals, the researchers of [14] implemented the interpolation methodology, Haar-based quadrature, and hybrid quadrature. The same investigators describe a method for the implementation of HOIs with fixed points in their subsequent study [19].

For the numerical method of the HFP and CPV integrals in the current assignment, we have employed the Newton Cotes type quadrature procedures [9], Haar-based, and hybrid quadrature procedures [1, 14, 18]. The intriguing aspect of applying these extrapolation methods to CPV and HFP integrals is that they eliminate the integral's singularity at x=0. On closely packed nodal points, the three techniques become more accurate. The correctness of the novel approaches from the perspective of HFP and CPV integrals is justified in solving a few experiments.

2 Quadrature: Newton Cotes Types

This section proposes a quadrature method of the Newton-Cote type for the numerical evaluation of CPV and HFP integrals.

2.1 CPV Integral

We create the $(4m-1)$ unit Newton cote form quadrature rule to calculate the CPV integral of the form:

$$I_c(g,\tau) = \int_{-a}^{b} \frac{g(x)}{x-\tau} dx, \tag{3}$$

where, the singular point of the integral is $a$and $a \in R^+$, $\tau = 0$.
We divide the closed interval $[-a,\ a]$into subintervals as $(4m-2)$points:

$$0, \pm\frac{a}{2m}, \pm\frac{3a}{2m}, \pm\frac{5a}{2m}, ..., \frac{(2m-1)\,a}{2m}. \tag{4}$$

Newton-cotes quadrature type formula for $(4m-1)$- points is represented as:

$$T_m\left(g\right) = W_{m0} g\left(0\right) + \sum_{i=1}^{2m-1} W_{mi} \left[g\left(\frac{la}{2m}\right) - g\left(-\frac{la}{2m}\right)\right]. \tag{5}$$

The weights W_{m0} and W_{mi}, are measured from the equation $AW = B$, by using the given nodal points in this formula where, $i = 1,\ 2,\ 3, ...,\ (2m-1)$. For all nodal points, we take$g(0) = 0$, to ignore the singularity at$x = 0$. The equation of moment, $AW = B$ in terms of matrix notation is given as:

$$\begin{bmatrix} 1 & 2 & ... & (2m-1) \\ 1^3 & 2^3 & ... & (2m-1)^3 \\ 1^5 & 2^5 & ... & (2m-1)^5 \\ ... & ... & ... & ... \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & ... & (2m-1) \\ 1^3 & 2^3 & ... & (2m-1)^3 \\ ... & ... & ... & ... \\ 1^{(2m-1)} & 2^{(2m-1)} & ... & (2m-1)^{(2m-1)} \end{bmatrix} \begin{bmatrix} W_{m1} \\ W_{m3} \\ . \\ . \\ W_{m(2m-1)} \end{bmatrix} = \begin{bmatrix} \frac{(2m-1)^1}{1} \\ \frac{(2m-1)^3}{3} \\ . \\ . \\ \frac{(2m-1)^{(2m-1)}}{(2m-1)} \end{bmatrix}. \tag{6}$$

For $m = 1, 2, 3, ..., 6$ Newton-cotes formula is derived and represented as:

$T_1(g) = 2\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right],$

$T_2(g) = 2.9778\left[g\left(\frac{a}{4}\right) - g\left(\frac{-a}{4}\right)\right] - 0.9156\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right] + 0.9511\left[g\left(\frac{3a}{4}\right) - g\left(\frac{-3a}{4}\right)\right],$

$T_3(g) = 8.3348\left[g\left(\frac{a}{6}\right) - g\left(\frac{-a}{6}\right)\right] - 8.06321\left[g\left(\frac{a}{3}\right) - g\left(\frac{-a}{3}\right)\right] + 6.0322\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right]$
$-2.0995\left[g\left(\frac{2a}{3}\right) - g\left(\frac{-2a}{3}\right)\right] + 0.7964\left[g\left(\frac{5a}{2}\right) - g\left(\frac{-5a}{2}\right)\right],$

$T_4(g) = 50.8200\left[g\left(\frac{a}{8}\right) - g\left(\frac{-a}{8}\right)\right] - 68.5079\left[g\left(\frac{a}{4}\right) - g\left(\frac{-a}{4}\right)\right] + 6.0322\left[g\left(\frac{3a}{8}\right) - g\left(\frac{-3a}{8}\right)\right]$
$-2.0995\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right] + 0.7964\left[g\left(\frac{5a}{8}\right) - g\left(\frac{-5a}{8}\right)\right] - 31.4820\left[g\left(\frac{3a}{4}\right) - g\left(\frac{-3a}{4}\right)\right]$
$+3.1484\left[g\left(\frac{7a}{8}\right) - g\left(\frac{-7a}{8}\right)\right],$

$T_5(g) = 433.0835\left[g\left(\frac{a}{10}\right) - g\left(\frac{-a}{10}\right)\right] - 647.4785\left[g\left(\frac{a}{5}\right) - g\left(\frac{-a}{5}\right)\right] + 599.7662\left[g\left(\frac{5a}{10}\right) - g\left(\frac{-5a}{10}\right)\right]$
$-400.9167\left[g\left(\frac{2a}{5}\right) - g\left(\frac{-2a}{5}\right)\right] + 202.0275\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right] - 76.2869\left[g\left(\frac{3a}{5}\right) - g\left(\frac{-3a}{5}\right)\right]$
$+21.5101\left[g\left(\frac{7a}{10}\right) - g\left(\frac{-7a}{10}\right)\right] - 4.1027\left[g\left(\frac{4a}{5}\right) - g\left(\frac{-4a}{5}\right)\right] + 0.6752\left[g\left(\frac{9a}{10}\right) - g\left(\frac{-9a}{10}\right)\right],$

$T_6(g) = 4241.563\left[g\left(\frac{a}{12}\right) - g\left(\frac{-a}{12}\right)\right] - 6665.610\left[g\left(\frac{a}{6}\right) - g\left(\frac{-a}{6}\right)\right] + 6675.656\left[g\left(\frac{a}{4}\right) - g\left(\frac{-a}{4}\right)\right]$
$-501.7547\left[g\left(\frac{a}{3}\right) - g\left(\frac{-a}{3}\right)\right] + 2959.769\left[g\left(\frac{5a}{12}\right) - g\left(\frac{-5a}{12}\right)\right] - 1386.985\left[g\left(\frac{a}{2}\right) - g\left(\frac{-a}{2}\right)\right]$
$+\left[g\left(\frac{7a}{12}\right) - g\left(\frac{-7a}{12}\right)\right] - 148.360\left[g\left(\frac{2a}{3}\right) - g\left(\frac{-2a}{3}\right)\right] + 32.505\left[g\left(\frac{3a}{4}\right) - g\left(\frac{-3a}{4}\right)\right]$
$-4.998\left[g\left(\frac{5a}{6}\right) - g\left(\frac{-5a}{6}\right)\right] + 0.6407\left[g\left(\frac{11a}{12}\right) - g\left(\frac{-11a}{12}\right)\right].$

2.2 HFP Integrals

To solve the following form of HFP integrals:

$$I_h(g,\tau) = H\int_{-a}^{a} \frac{g(x)}{(x-\tau)^2}dx. \tag{7}$$

Consider that$g(z)$ is the analytic continuation of $g(x)$in the disc:

$$S = \{x \in C : |z| \le r : r > a\}, \tag{8}$$

such that

$$g(z) = g(x) : \ \forall\, x \in [-a, \ a]. \tag{9}$$

Integral $I_h(g,)$ for $\tau = 0$ can be written as:

$I_h(g,\tau) = \int_{-a}^{a} \frac{\left(\frac{g(x)-k_1}{x}\right)}{x}dx + \int_{-a}^{a} \frac{k_1}{x^2}dx,$ where, $k_1 = res_{x=0\left\{\frac{g(x)}{x}\right\}},$

$$= \int_{-a}^{a} \frac{h(x)}{x}dx + \int_{-a}^{a} \frac{1}{x^2}dx = K + L. \tag{10}$$

, singular integral represented by L, which can be calculated with the help of calculus fundamental theorem and similarly by using the rules from T_1 to T_6 we can calculate the value of K, where$K$represents CPV integral.

3 Quadrature Rules of Multi-Resolution

We gave a brief overview of multi-resolution quadrature procedures in this section, including hybrid functions and Haar wavelets of order m [1, 16]. In [1, 15], a thorough explanation of these quadrature principles is provided.

3.1 Quadrature Based on Haar Wavelets

An overview and detailed explanation of the Haar wavelets functions is provided in [15]. For the numerical assessment of the HFP and CPV integrals, which are given by:

$$\int_c^d g(x) \approx \frac{(d-c)}{N} \sum_{i=1}^{N} \left(c + \frac{(d-c)\left(l-\frac{1}{2}\right)}{N} \right), \tag{11}$$

where, $N = 2m$.

3.2 Quadrature of Hybrid-Based Functions

In this part, we provide an in-depth discussion of quadrature, based on hybrid functions for the numerical assessment of the HFP and CPV integrals. The equations of order $m = 1, 2, 3, ..., 10$ for hybrid functions are described extensively and established in [1, 15]. We employ the quadrature, based on hybrid functions of orders$m = 2$ and $m = 8$ in the present study. Formula for $m = 6$ is given by:

$$\int_c^d g(x)\, dx = \frac{d-c}{1280n} \sum_{i=1}^{n} \left[\begin{array}{c} \left(247g\left(c+\frac{(d-c)(12l-11)}{N}\right)\right) + \left(139g\left(c+\frac{(d-c)(12l-9)}{N}\right)\right) \\ + \left(254g\left(c+\frac{(d-c)(12l-7)}{N}\right)\right) + \left(254g\left(c+\frac{(d-c)(12l-5)}{N}\right)\right) \\ + \left(139g\left(c+\frac{(d-c)(12l-3)}{N}\right)\right) + \left(247g\left(c+\frac{(d-c)(12l-1)}{N}\right)\right) \end{array} \right] \tag{12}$$

where, $N = 2n$.
Similar to this, the following is the formula for quadrature, based on hybrid function of order $m = 8$:

$$\int_c^d g(x)\, dx = \frac{d-c}{1935360n} \sum_{i=1}^{n} \left[\begin{array}{c} \left(295627g\left(c+\frac{(d-c)(16l-15)}{N}\right)\right) + \left(7139g\left(c+\frac{(d-c)(16l-13)}{N}\right)\right) \\ + \left(471771g\left(c+\frac{(d-c)(16l-11)}{N}\right)\right) + \left(128953g\left(c+\frac{(d-c)(16l-9)}{N}\right)\right) \\ + \left(128953g\left(c+\frac{(d-c)(16l-7)}{N}\right)\right) + \left(471771g\left(c+\frac{(d-c)(16l-7)}{N}\right)\right) \\ + \left(7139g\left(c+\frac{(d-c)(16l-3)}{N}\right)\right) + \left(295627g\left(c+\frac{(d-c)(16l-1)}{N}\right)\right) \end{array} \right] \tag{13}$$

where, $N = 16n$.

4 Numerical Results

To determine whether the suggested approaches are accurate, certain test issues have been addressed. In this regard, there are two each of the HFP and CPV test integrals are included. The precise answers to the first and fourth test issues are obtained from [9], but the precise answers to the second and third test questions are discovered employing a hybrid function of order $m = 8$at upper nodal points. All test issues include calculations for the absolute error L_{ab}.

Test Problem 1. Following is the considered CPV integral from reference [9]:

$$I_c\left(e^x, \tau\right) = p \int_{-1}^{1} \frac{e^x}{(x-\tau)} dx. \tag{14}$$

In the present interval of integration, the integrand has a singularity at time$\tau = 0$. Because of the existence of a singular point in the interval [-1, 1], conventional quadrature methods such as the Simpson rule, etc. are unable to determine these sorts of integrals. HY6, NC, HY8, and HW calculate the integral. Table 1 displays the outcomes with regard to absolute error. Additionally, the outcomes of the suggested techniques are contrasted with those of the Newton-Cotes formula [9]. The table demonstrates that even though the outcomes of HY6 and HY8 are superior to the findings determined in reference [9], also the findings of NC and HW are comparable at the same nodal positions.

Test Problem 2. Computing of the CPV integral numerically:

$$I_c\left(e^x, \tau\right) = p \int_{-1}^{1} \frac{e^x}{(x-\tau)} dx. \tag{15}$$

Table 1: Test of first problem for L_{ab} of NC, HY6, HY8 and HW

N	HY8	HY6	HW	NC	NC[9]
3	4.55 e^{-12}	6.23 e^{-09}	1.91 e^{-03}	3.01 e^{-02}	3.00 e^{-02}
7	5.40 e^{-13}	3.95 e^{-11}	7.48 e^{-06}	9.30 e^{-06}	9.31 e^{-06}
11	5.57 e^{-13}	3.13 e^{-12}	2.92 e^{-08}	2.59 e^{-10}	2.60 e^{-10}
15	5.37 e^{-13}	9.47 e^{-13}	1.14 e^{-10}	9.42 e^{-13}	MP
19	5.36 e^{-13}	6.42 e^{-13}	9.89 e^{-13}	5.53 e^{-06}	
23	5.21 e^{-13}	5.75 e^{-13}	4.46 e^{-13}	4.88 e^{-01}	

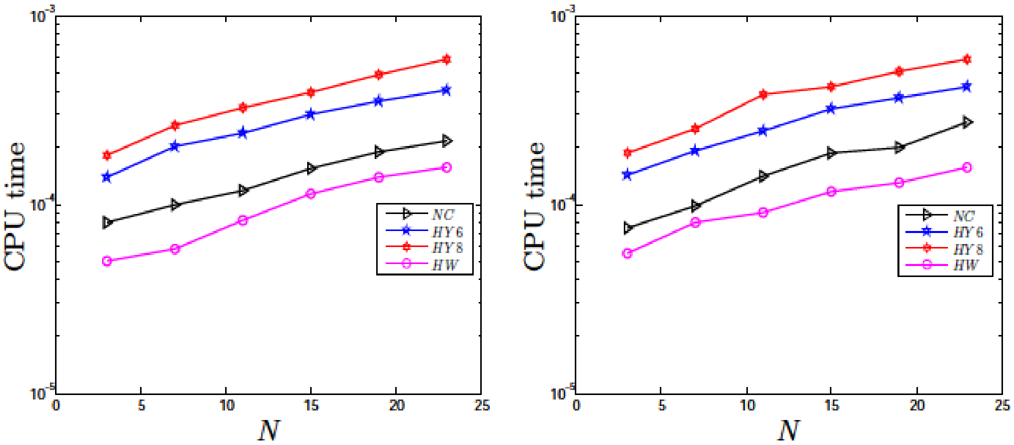

Figure 1: Comparison graph of HY8, NC, HY6, and HW for test problem 3 (Left) and 4 (Right), where, N = 3, 7,. . . , 23. [CPU time is measured in seconds]

By utilizing the suggested procedures NC, HY6, HY8, and HW integral is evaluated. Table 2 and Fig. 1b, correspondingly, compare lab and CPU time (measured in seconds). The suggested technique HY8 provides the precision of order $O(10^{-14})$as indicated in the table, whereas the accuracy of all other approaches degrades up to $O(10^{-12})$ for small nodal points, such as $N = 23$. Because of the NC procedure's instability at higher nodal positions, its precision is subject to oscillation. With dense nodes, precision is improved by the HW, HY6, and HY8. As a result, quadrature based on hybrid functions and Haar wavelets are reliable at higher nodes.These three approaches are effective even at higher nodal points, as demonstrated in Fig. 1b.

Table 2: Test of second problem for L_{ab} of NC, HY6, HY8 and HW

N	HY6	HY8	HW	NC	N
3	2.89 e^{-07}	3.10 e^{-10}	1.60 e^{-02}	2.44 e^{-01}	3
7	1.82 e^{-09}	3.46 e^{-13}	6.27 e^{-05}	4.07 e^{-04}	7
11	1.21 e^{-10}	1.90 e^{-14}	2.45 e^{-07}	2.89 e^{-08}	11
15	1.88 e^{-11}	8.43 e^{-14}	9.58 e^{-10}	9.84 e^{-12}	15
19	4.56 e^{-12}	1.15 e^{-14}	3.79 e^{-12}	6.45 e^{-06}	19
23	1.44 e^{-12}	2.39 e^{-14}	1.42 e^{-13}	5.12 e^{-01}	23

Test Problem 3. Following is the considered CPV integral:

$$I_h\left(1-e^{2x},\tau\right) = H\int_{-2}^{2} \frac{\left(1-e^{2x}\right)}{\left(x-\tau\right)^2}dx. \tag{16}$$

In the present interval of integration, the integrand has a singularity at time $\tau = x$. Because of the existence of a second-order singularity at the point $\tau = x$, conventional quadrature methods such as the Gaussian quadrature etc, etc. are unable to determine these sorts of integrals. The suggested techniques analyze the integral. The outcomes for the L_{ab} are displayed in Tab. 3. The table demonstrates that for large grid points, all three approaches provide a high level of reliability. In Fig. 2 a, the CPU time comparison of all approaches is shown. It is evident from the graphic that finding HFP integrals takes a comparable amount of time as finding CPV integrals. This demonstrates that the sequence of singularity has no bearing on the effectiveness and correctness of the method approaches.

Table 3: Test of the third problem for L_{ab} of NC, HY6, HY8 and HW

N	N C	HY 6	HW	HY 8
3	2.05 e^{-03}	7.56 e^{-05}	1.45 e^{-01}	6.64 e^{-07}
7	6.34 e^{-07}	5.29 e^{-07}	5.75 e^{-04}	8.69 e^{-10}
11	4.65 e^{-10}	3.57 e^{-08}	2.24 e^{-06}	2.38 e^{-11}
15	3.52 e^{-12}	5.58 e^{-09}	8.78 e^{-09}	2.02 e^{-12}
19	1.37 e^{-06}	1.35 e^{-09}	9.14 e^{-11}	3.10 e^{-13}
23	7.94 e^{-01}	4.31 e^{-10}	8.90 e^{-12}	5.32 e^{-14}

Test Problem 4. Following is the computation integral of HFP [9]:

$$I_h\left(e^{-x}\sin x, \tau\right) = H\int_{-1}^{1}\frac{\left(e^{-x}\sin x\right)}{\left(x-\tau\right)^2}dx. \tag{17}$$

Utilizing the novel approaches NC, HW, HY 6 and HY 8, the integral is assessed. Table 4 displays the mean errors for N=2,3,7,. . . ,23. Findings from the new methodologies are contrasted with NC [9]. The table demonstrates that for the same nodal positions, the new approaches provide more precise results than the old approach [9]. The table 4 (right) comparison of CPU time is displayed. It is evident from the entire debate that the latest techniques are more reliable and precise than the current literature.

Table 4: L_{ab} of HY6, NC, HY8, and HW for test problem 4

N	NC	HW	HY6	HY8	NC [9]
3	3.05 e^{-03}	2.28 e^{-04}	2.10 e^{-09}	1.12 e^{-11}	3.00 e^{-03}
7	7.31 e^{-07}	9.02 e^{-07}	1.40 e^{-11}	4.79 e^{-14}	7.30 e^{-07}
11	3.67 e^{-11}	3.52 e^{-09}	9.72 e^{-13}	1.59 e^{-14}	3.70 e^{-11}
15	2.57 e^{-14}	1.37 e^{-11}	1.77 e^{-13}	3.55 e^{-14}	M P
19	5.38 e^{-06}	8.81 e^{-14}	6.70 e^{-14}	3.88 e^{-14}	
2	4.84 e^{-01}	3.78 e^{-13}	4.26 e^{-14}	5.24 e^{-14}	

5 Conclusion

For the numerical assessment of singular and hyper-singular integrals in this study, quadrature based on Newton Cote type, Haar, and hybrid functions is employed. The CPV and HFP integrals are some of these integrals. The suggested techniques can be used to evaluate integrals with singularity at $\tau = 0$. Numerical testing demonstrates the new techniques' accuracy and effectiveness.

Nomenclature

Symbols	**Description**
HY6	Hybrid based quadrature of order
HFP	Hadamard finite part integral
HW	Haar wavelets based quadrature
NC	Newton cotes type quadrature
CPV	6Cauchy principle value integral
HY8	Hybrid based quadrature of order 8
^{L}ab	Absolute Error
MP	Maximum precision

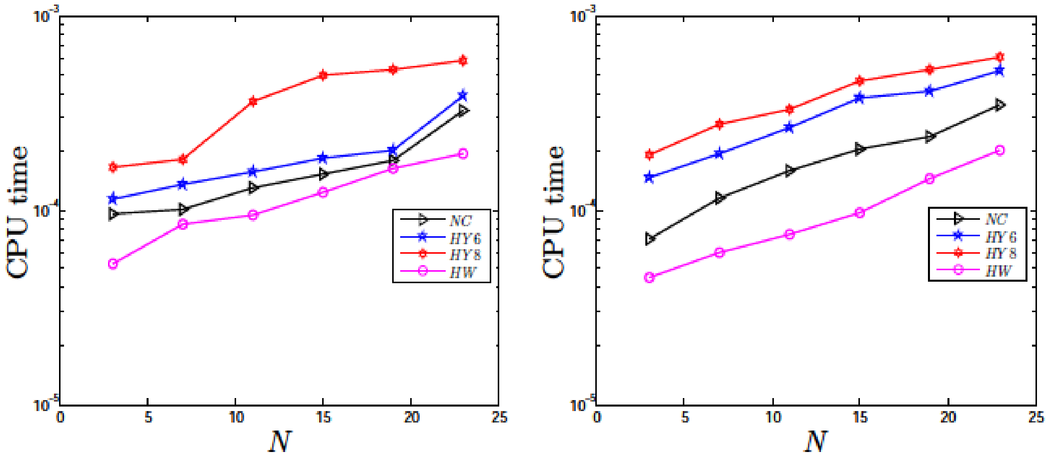

Figure 2: Comparison graph of HY8, NC, HY6, and HW for test problem 3 (Left) and 4 (Right), where, $N = 3, 7, \ldots, 23$

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