

Unsteady Flow of Casson Fluid between Two Side Walls over an Infinite Plate

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ABSTRACT

In the present study, the velocity field corresponding to the unsteady flow of Casson fluid between two parallel side walls perpendicular to an infinite plate that exerts an oscillating shear stress to the fluid is determined with the help of Fourier sine and cosine transforms. Exact solutions are obtained and given in simple form as a combination of transient and steady-state solutions, which satisfy the governing equation and all imposed initial and boundary conditions. These solutions are reduced to similar solutions corresponding to the motion over an infinite plate, when $h \rightarrow \infty$, and Newtonian fluid by making Casson parameter $\beta \rightarrow \infty$. Furthermore, the effect of side walls on the motion of fluid, the time which is required to attain the steady state and the distance between the side walls for which their presence can be ignored are graphically determined.

1 Introduction

Many materials are used in modern engineering whose flow characteristics are not properly explained with the help of the Newtonian fluid model. So, due to this reason, the study of non-Newtonian fluid becomes essential. Besides this non-Newtonian fluids are those in which the relation between the stress and strain are non-linear. Examples of non-Newtonian fluids are toothpaste, polymer melts, chocolate, jelly, soap, greases and oils, etc. Non-Newtonian fluids are used widely in the fields of food industries, chemical engineering, medicine sciences, petroleum, and many other fields [1]. Several engineers and scientists are working on non-Newtonian fluids. The physical properties of non-Newtonian fluid are not easy just like Newtonian fluid. So, this is the reason that a single constitutive equation cannot explain easily all the physical properties of non-Newtonian fluid. Because of this fact numerous models or constitutive equations are developed for different types of flow of non-Newtonian fluid. Some of them are Jeffery fluid [2], Second-grade fluid [3], Maxwell fluid [4], Viscoplastic fluid [5], Bingham plastic [6], Brinkman type fluid [7] and Walter-B fluid [8], and Oldroyd-B fluid [9] models.

Besides these models, there is another model recently the most popular one, known as the Casson model. In 1959 the Casson fluid model was introduced by Casson, for the prediction of the flow behavior of pigment-oil suspensions [10]. Casson fluid is one of the pseudo-plastic fluids that means shear thinning fluids at low shear rate the shear thinning fluid is more viscous than the Newtonian fluid and at a high shear rate it is less viscous [11]. Later on, Casson fluid is studied by many researchers for different flow configurations and situations. Among them, a solution for the unsteady flow of Casson fluid passed over a semi-infinite vertical plate with thermal and hydrodynamic slip conditions is obtained by Rao *et al.* [12]. Mustafa *et al.* [13] consider the unsteady flow of Casson fluid past over a moving flat plate with a heat transfer effect. Hussanan *et al.* [14] investigate the boundary layer flow of Casson fluid passed an oscillating vertical plate with Newtonian heating. Raju [15] has studied the effect of an induced magnetic field on the stagnation flow of Casson fluid. A closed-form solution for the boundary layer flow

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of Casson fluid over a permeable shrinking / stretching sheet without and with an extended magnetic field was obtained by Bathacharyya *et al.* [16-17]. The pioneering work on the closed-form solution for free convection and electrically conducting the flow of Casson fluid over an oscillating vertical plate passing through a porous medium is studied by Khalid *et al* [18]. Makanda [19-20] has discussed the effect of radiation as well as the chemical reaction of Casson fluid flow.

Recently, the interest of scholars in the flow of fluids between side walls, over an infinite plate using different boundary conditions has significantly increased. The references [21-30] can be cited for the motion of the fluid between two side walls over an infinite plate that exerts oscillating or constant shear stress on the fluid. The present article has been undertaken to determine the starting solutions for the flow of Casson fluid between two side walls over an infinite plate that exerts an oscillating shear stress on the fluid.

2 Mathematical Analysis of the Problem

Let us consider the unsteady two-dimensional flow of Casson fluid between two parallel side walls normal to the infinite plate and initially, both, the fluid and infinite plate are stationary. After a time $t = 0^+$ the plate exerts an oscillating shear stress of the form $\{f \cos \omega t$ or $f \sin \omega t\}$ to the fluid and as a result, the fluid moved gradually as shown in Fig. 1. Its velocity $\vec{\mathbf{V}}$ and extra stress \mathbf{S} are of the form:

$$\vec{\mathbf{V}} = u(y, z, t) \mathbf{i}, \quad \mathbf{S} = \mathbf{S}(y, z, t), \quad (1)$$

where \mathbf{i} represents unit vector along the x -axis of the Cartesian coordinate system x, y and z . The rheological equation for the incompressible flow of Casson fluid is given by [31]:

$$\tau_{ij} = \left\{ \begin{array}{ll} 2 \left(\mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, & \pi_c < \pi \\ 2 \left(\mu_B + \frac{p_y}{\sqrt{2\pi_c}} \right) e_{ij}, & \pi_c > \pi \end{array} \right\}, \quad (2)$$

here, π represents the product of the component of the rate of deformation with itself, μ_B stands for plastic dynamic viscosity, e_{ij} denotes $(ij)^{th}$ components of the deformation rate, p_y is the yield stress of the fluid and π_c represents the critical value of this product based on the non-Newtonian model. For such a flow, the constraint of incompressibility is

automatically satisfied and the governing equation after using equations (1) and (2), is

$$\frac{\partial u(y, z, t)}{\partial t} = \nu \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u(y, z, t)}{\partial y^2} + \frac{\partial^2 u(y, z, t)}{\partial z^2} \right), \quad y, t > 0, \quad z \in [0, d], \quad (3)$$

where $\nu = \frac{\mu_B}{\rho}$ denotes kinematic viscosity, d represents distance between the walls, $\beta = \mu_B \sqrt{2\pi_c} / p_y$ is the Casson parameter and the appropriate initial and boundary conditions are:

$$\begin{aligned} u(y, z, 0) &= 0, \quad y > 0, \quad z \in [0, d], \\ u(y, 0, t) &= u(y, d, t) = 0 \text{ for } y, t > 0, \end{aligned} \quad (4)$$

$$\tau(0, z, t) = \mu \left(1 + \frac{1}{\beta} \right) \frac{\partial u(y, z, t)}{\partial y} \Big|_{y=0} = f \cos(\omega t) \text{ or } f \sin(\omega t), \quad t > 0, \quad z \in (0, d), \quad (5)$$

$$\frac{\partial u(y, z, t)}{\partial y}, \quad u(y, z, t) \rightarrow 0, \quad \text{as } y \rightarrow \infty, \quad \text{for } t > 0, \quad z \in [0, d]. \quad (6)$$

3 Procedure for Solution of Problem

3.1 The Case $\tau(0, z, t) = f \sin(\omega t)$

We consider the flow of Casson fluid between two parallel side walls over an infinite plate situated at the plane $z = 0$ and $z = d$, the flow being confined in (x, z) -plane. At time $t = 0^+$, the plate applies an oscillating shear stress to the fluid.

$$\tau(0, z, t) = \frac{\partial u(y, z, t)}{\partial y} \Big|_{y=0} = \frac{f}{\mu} \cos(\omega t) \text{ or } \frac{f}{\mu} \sin(\omega t) \text{ for } z \in (0, d) \text{ and } t > 0. \quad (7)$$

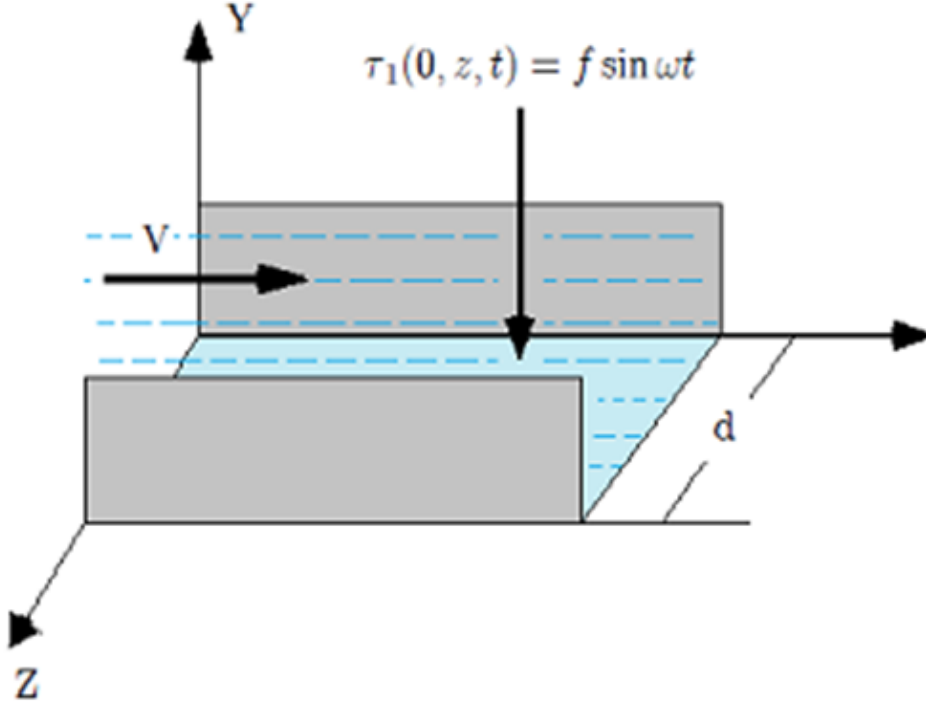


Figure 1: Procedure for Solution of Problem

Both sides of the governing Eq. (3) are multiplying by $\sqrt{\frac{2}{\pi}} \cos(\zeta y) \sin(\eta_n z)$, where $\eta_n = \frac{n\pi}{d}$, and then integrating the obtained result with respect to z and y from 0 to d and from 0 to ∞ , respectively, and keeping in mind the boundary and initial conditions (4), (5), and (6), we get the following differential equation.

$$\frac{\partial u_{cn}(\zeta, t)}{\partial t} + \nu \left(1 + \frac{1}{\beta}\right) (\zeta^2 + \eta_n^2) u_{cn}(\zeta, t) = \sqrt{\frac{2}{\pi}} \frac{\nu f}{\mu} \left[\frac{(-1)^n - 1}{\eta_n} \right] \sin(\omega t), \quad (8)$$

where the definition of Fourier cosine and sine transforms of $u(y, z, t)$ is:

$$u_{cn}(\zeta, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^d u(y, z, t) \cos(y\zeta) \sin(\eta_n z) dz dy, \quad n = 1, 2, 3, \dots, \quad (9)$$

and it satisfies the following initial condition:

$$u_{cn}(\zeta, 0) = 0 \text{ for } \zeta > 0 \text{ and } n = 1, 2, 3, \dots, \quad (10)$$

Eq. (8) can be expressed an ordinary differential equation in t , for each fixed ζ . By using the initial condition Eq. (10), we get the solution of Eq. (8) as:

$$\begin{aligned} u_{cn}(\zeta, t) = & \frac{f}{\mu \left(1 + \frac{1}{\beta}\right)^2} \sqrt{\frac{2}{\pi}} \frac{(-1)^n - 1}{\eta_n} \times \left[\frac{(\zeta^2 + \eta_n^2) \sin(\omega t) - \frac{\omega}{\nu} \frac{1}{\left(1 + \frac{1}{\beta}\right)} \cos(\omega t)}{\left(\frac{\omega}{\nu} \frac{1}{\left(1 + \frac{1}{\beta}\right)}\right)^2 + (\zeta^2 + \eta_n^2)^2} \right] \\ & + \frac{f\omega}{\nu \mu \left(1 + \frac{1}{\beta}\right)} \sqrt{\frac{2}{\pi}} \left[\frac{(-1)^n - 1}{\eta_n} \right] \left[\frac{1}{\left(\frac{\omega}{\nu} \frac{1}{\left(1 + \frac{1}{\beta}\right)}\right)^2 + (\zeta^2 + \eta_n^2)^2} \right] \\ & \times \exp \left[-\nu \left(1 + \frac{1}{\beta}\right) (\zeta^2 + \eta_n^2) t \right]. \end{aligned} \quad (11)$$

The expression for velocity $u_s(y, z, t)$ is obtained by applying the Fourier inversion formulae [32, 33] on

Eq. (11).

$$\begin{aligned}
u_s(y, z, t) &= \frac{8f}{\mu\pi d} \sum_{n=1}^{\infty} \frac{\sin(\eta_m z)}{\eta_m} \frac{1}{(1+\frac{1}{\beta})} \int_0^{\infty} \frac{\frac{\omega}{\nu} (1+\frac{1}{\beta}) \cos(\omega t) - (\zeta^2 + \eta_m^2) \sin(\omega t)}{(\zeta^2 + \eta_m^2)^2 + (\frac{\omega}{\nu})^2 (1+\frac{1}{\beta})^2} \cos(y\zeta) d\zeta \\
&- \frac{8f}{\mu\pi d} \sum_{n=1}^{\infty} \frac{\sin(\eta_m z)}{\eta_m} \frac{1}{(1+\frac{1}{\beta})} \left(\frac{\omega}{\nu}\right) \int_0^{\infty} \frac{\cos(y\zeta) e^{-\nu(1+\frac{1}{\beta})(\zeta^2 + \eta_m^2)t}}{(\zeta^2 + \eta_m^2)^2 + (\frac{\omega}{\nu})^2 (1+\frac{1}{\beta})^2} d\zeta,
\end{aligned} \tag{12}$$

setting $d = 2h$, $m = 2n - 1$, and varying the origin of the coordinate system, taking $z = z' + h$ and neglecting the prime notation, we get:

$$\begin{aligned}
u_s(y, z, t) &= \frac{4f}{\mu\pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{1}{(1+\frac{1}{\beta})} \\
&\times \int_0^{\infty} \frac{(\zeta^2 + \varepsilon_m^2) \sin(\omega t) - \frac{\omega}{\nu} (1+\frac{1}{\beta}) \cos(\omega t)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu})^2 (1+\frac{1}{\beta})^2} \cos(y\zeta) d\zeta \\
&+ \frac{4f}{\mu\pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{1}{(1+\frac{1}{\beta})} \left(\frac{\omega}{\nu}\right) \int_0^{\infty} \frac{\cos(y\zeta) e^{-\nu(1+\frac{1}{\beta})(\zeta^2 + \varepsilon_m^2)t}}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu})^2 (1+\frac{1}{\beta})^2} d\zeta,
\end{aligned} \tag{13}$$

where $\varepsilon_m = (2n - 1) \frac{\pi}{2h}$, now let $N = \frac{1}{1+\frac{1}{\beta}}$, and some identities used in this problem are:

$$\int_0^{\infty} \frac{(\zeta^2 + \varepsilon_m^2) \cos(\zeta y)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} d\zeta = \frac{\pi}{2} \frac{D_m \cos(yC_m) - C_m \sin(yC_m)}{C_m^2 + D_m^2} e^{-yC_m}, \tag{14}$$

$$\left(\frac{\omega}{\nu}\right) N \int_0^{\infty} \frac{\cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} d\zeta = \frac{\pi}{2} \frac{C_m \cos(yC_m) + D_m \sin(yC_m)}{C_m^2 + D_m^2} e^{-yD_m}, \tag{15}$$

where $2C_m^2 = \sqrt{\varepsilon_m^4 + (\frac{\omega}{\nu} N)^2} - \varepsilon_m^2$ and $2D_m^2 = \sqrt{\varepsilon_m^4 + (\frac{\omega}{\nu} N)^2} + \varepsilon_m^2$, by using the above identities we get the following expression for the velocity field.

$$\begin{aligned}
u_s(y, z, t) &= \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{e^{-yD_m}}{C_m^2 + D_m^2} [D_m \sin(\omega t - yC_m) - C_m \cos(\omega t - yC_m)] \\
&+ \frac{4f}{\pi\mu h} \left(\frac{\omega}{\nu}\right) N \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{\cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t} d\zeta.
\end{aligned} \tag{16}$$

Eq. (16) represents the exact solution, which is the combination of steady and transient solutions. Eq. (16), satisfies all the imposed initial and boundary conditions and they describe the motion of the fluid sometime after their initiation. The transient solution disappears after that time and the exact solution then tends to the steady state solution. Both, transient and steady-state solutions are given in Eq. (17) and Eq. (18) respectively,

$$u_{st}(y, z, t) = \frac{4f}{\pi\mu h} \left(\frac{\omega}{\nu}\right) N \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{\cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t} d\zeta, \tag{17}$$

and

$$u_{ss}(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \left[\frac{D_m \sin(\omega t - C_m y) - C_m \cos(\omega t - C_m y)}{C_m^2 + D_m^2} \right] e^{-yD_m}. \tag{18}$$

There are two types of non-trivial shear stresses that are used in this problem (i) shear stress on the fluid applied by the bottom wall and (ii) shear stress due to side walls, i.e. $\tau(y, z, t) = S_{xy}(y, z, t)$ and $S_{xz}(y, z, t)$ respectively. $\tau(y, z, t) = S_{xy}(y, z, t)$ can be calculated as follows. By taking the derivative of Eq. (16) with respect to y'' , we get:

$$\begin{aligned}
\tau_s(y, z, t) &= \frac{2f}{h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} e^{-yD_m} \sin(\omega t - C_m y) \\
&+ \frac{4f}{\pi h} \left(\frac{\omega}{\nu}\right) N \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{\sin(y\zeta) e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t}}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} d\zeta.
\end{aligned} \tag{19}$$

3.2 The Case $\tau(0, z, t) = f \cos(\omega t)$

Using the above procedure, we find the corresponding starting solution for cosine oscillation.

$$u_c(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{e^{-yD_m}}{C_m^2 + D_m^2} \times [C_m \sin(\omega t - yC_m) + D_m \cos(\omega t - yC_m)] - \frac{4f}{\pi \mu h} \times \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \times \int_0^{\infty} \frac{(\zeta^2 + \varepsilon_m^2) \cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t} d\zeta. \quad (20)$$

Eq. (20) represents the exact solution, which is the combination of steady and transient solutions. Eq. (20) satisfies all the imposed initial and boundary conditions and they describe the motion of the fluid sometime after their initiation. The transient solution disappears after that time and the exact solution then tend to the steady state solution. Both, transient and steady-state solutions are given in Eq. (21) and Eq. (22) respectively.

$$u_{ct}(y, z, t) = \frac{4f}{\pi \mu h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{(\zeta^2 + \varepsilon_m^2) \cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t} d\zeta, \quad (21)$$

and

$$u_{cs}(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{e^{-yD_m}}{C_m^2 + D_m^2} [C_m \sin(\omega t - yC_m) + D_m \cos(\omega t - yC_m)]. \quad (22)$$

There are two types of non-trivial shear stresses that are used in this problem (i) shear stress on the fluid applied by the bottom wall and (ii) shear stress due to side walls, i.e. $\tau(y, z, t) = S_{xy}(y, z, t)$ and $S_{xz}(y, z, t)$ respectively. $\tau(y, z, t) = S_{xy}(y, z, t)$ can be calculated as follows. By taking the derivative of Eq. (20), with respect to "y", we get:

$$\tau_c(y, z, t) = \frac{2f}{h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} e^{-yD_m} \cos(\omega t - C_m y) - \frac{4f}{\pi h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{\zeta(\zeta^2 + \varepsilon_m^2) \sin(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + (\frac{\omega}{\nu} N)^2} e^{-\frac{\nu}{N}(\zeta^2 + \varepsilon_m^2)t} d\zeta. \quad (23)$$

The velocities $u_s(y, z, t)$, $u_c(y, z, t)$ and the shear stresses give us the exact solutions for the motion of the Casson fluid sometime after its initiation. The transient part of the solution vanishes after that initiation time and we get the steady-state solutions. In addition, these solutions are free from initial conditions and periodic in time. Furthermore, they satisfy the governing equations as well as the boundary conditions.

4 Special Cases

4.1 Flow over an Infinite Plate

By taking $h \rightarrow \infty$, the general solutions (16), (19), (20) and (23), becomes:

$$u_s(y, t) = \frac{f}{\mu} \sqrt{\frac{\nu}{\omega}} N e^{-y\sqrt{\frac{\omega}{2\nu}N}} \sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}N} + \frac{3\pi}{4}\right) - \frac{2f}{\mu\pi} \left(\frac{\omega}{\nu}\right) N \int_0^{\infty} \frac{\cos(y\zeta)}{\zeta^4 + (\frac{\omega}{\nu}N)^2} e^{-\frac{\nu}{N}\zeta^2 t} d\zeta, \quad (24)$$

where, $\omega \neq 0$

$$\tau_s(y, t) = f e^{-y\sqrt{\frac{\omega}{2\nu}N}} \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}N}\right) + \frac{2f}{\pi} \left(\frac{\omega}{\nu}\right) N \int_0^{\infty} \frac{\zeta \sin(y\zeta)}{\zeta^2 + (\frac{\omega}{\nu}N)^2} e^{-\frac{\nu}{N}\zeta^2 t} d\zeta, \quad (25)$$

$$u_c(y, t) = \frac{f}{\mu} \sqrt{\frac{\nu}{\omega}} N e^{-y\sqrt{\frac{\omega}{2\nu}N}} \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}N} + \frac{3\pi}{4}\right) + \frac{2f}{\mu\pi} \int_0^{\infty} \frac{\zeta^2 \cos(y\zeta)}{\zeta^4 + (\frac{\omega}{\nu}N)^2} e^{-\frac{\nu}{N}\zeta^2 t} d\zeta, \quad (26)$$

where, $\omega \neq 0$, respectively,

$$\tau_c(y, t) = f e^{-y\sqrt{\frac{\omega}{2\nu}N}} \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}N}\right) - \frac{2f}{\pi} \int_0^{\infty} \frac{\zeta^3 \sin(y\zeta)}{\zeta^2 + (\frac{\omega}{\nu}N)^2} e^{-\frac{\nu}{N}\zeta^2 t} d\zeta. \quad (27)$$

4.2 Newtonian Fluid $\beta \rightarrow \infty$

By making $\beta \rightarrow \infty$ into Eqs. (16), (19), (20) and (23), we recover the same solutions for velocity and shear stress, as obtained by Fetecau *et al.* [34], (Eqs. (14), (17), (18) and (19)):

$$u_{sN}(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{e^{-yD_m}}{C_m^2 + D_m^2} [D_m \sin(\omega t - yC_m) - C_m \cos(\omega t - yC_m)] \\ + \frac{4f}{\pi \mu h} \left(\frac{\omega}{\nu}\right) \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \times \int_0^{\infty} \frac{\cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + \left(\frac{\omega}{\nu}\right)^2} e^{-\nu(\zeta^2 + \varepsilon_m^2)t} d\zeta, \quad (28)$$

and

$$\tau_{sN}(x, y, z) = \frac{2f}{h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} e^{-yD_m} \sin(\omega t - C_m y) \\ + \frac{4f}{\pi h} \left(\frac{\omega}{\nu}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} \int_0^{\infty} \frac{\sin(y\zeta) e^{-\nu(\zeta^2 + \varepsilon_m^2)t}}{(\zeta^2 + \varepsilon_m^2)^2 + \left(\frac{\omega}{\nu}\right)^2} d\zeta, \quad (29)$$

Similarly,

$$u_{cN}(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \frac{e^{-yB_m}}{C_m^2 + D_m^2} [C_m \sin(\omega t - yC_m) + D_m \cos(\omega t - yC_m)] \\ - \frac{4f}{\pi \mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\varepsilon_m z)}{\varepsilon_m} \times \int_0^{\infty} \frac{(\zeta^2 + \varepsilon_m^2) \cos(y\zeta)}{(\zeta^2 + \varepsilon_m^2)^2 + \left(\frac{\omega}{\nu}\right)^2} e^{-\nu(\zeta^2 + \varepsilon_m^2)t} d\zeta, \quad (30)$$

and

$$\tau_{cN}(y, z, t) = \frac{2f}{h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} e^{-yD_m} \cos(\omega t - C_m y) - \frac{4f}{\pi h} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos(\varepsilon_m z)}{\varepsilon_m} \\ \times \int_0^{\infty} \frac{\zeta(\zeta^2 + \varepsilon_m^2) \sin(y\zeta) e^{-\nu(\zeta^2 + \varepsilon_m^2)t}}{(\zeta^2 + \varepsilon_m^2)^2 + \left(\frac{\omega}{\nu}\right)^2} d\zeta. \quad (31)$$

4.3 Flow of Newtonian Fluid over an Infinite Plate

By making $h \rightarrow \infty$, into Eqs. (28–31), we recover the same solutions for velocity and shear stress of Newtonian fluid, as obtained by Fetecau *et al.* [34], (Eqs. (20), (21), (22) and (23)):

$$u_{sN}(y, t) = \frac{f}{\mu} \sqrt{\frac{\nu}{\omega}} e^{-y\sqrt{\frac{\omega}{2\nu}}} \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}} + \frac{3\pi}{4}\right) - \frac{2f}{\mu\pi} \frac{\omega}{\nu} \int_0^{\infty} \frac{\cos(y\zeta)}{\zeta^4 + \left(\frac{\nu}{\omega}\right)^2} e^{-\nu\zeta^2 t} d\zeta, \quad (32)$$

$$\tau_{sN}(y, t) = f e^{-y\sqrt{\frac{\omega}{2\nu}}} \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) + \frac{2f\omega}{\pi\nu} \int_0^{\infty} \frac{\zeta \sin(y\zeta)}{\zeta^2 + \left(\frac{\nu}{\omega}\right)^2} e^{-\nu\zeta^2 t} d\zeta, \quad (33)$$

similarly,

$$u_{cN}(y, t) = \frac{f}{\mu} \sqrt{\frac{\nu}{\omega}} e^{-y\sqrt{\frac{\omega}{2\nu}}} \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}} + \frac{3\pi}{4}\right) + \frac{2f}{\mu\pi} \int_0^{\infty} \frac{\zeta^2 \cos(y\zeta)}{\zeta^4 + \left(\frac{\nu}{\omega}\right)^2} e^{-\nu\zeta^2 t} d\zeta, \quad (34)$$

$$\tau_{cN}(y, t) = f e^{-y\sqrt{\frac{\omega}{2\nu}}} \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) - \frac{2f}{\pi} \int_0^{\infty} \frac{\zeta^3 \sin(y\zeta)}{\zeta^2 + \left(\frac{\nu}{\omega}\right)^2} e^{-\nu\zeta^2 t} d\zeta. \quad (35)$$

5 Numerical Results and Discussion

This paper describes about the study of the unidirectional and two-dimensional flow of Casson fluid between two parallel side walls perpendicular to an infinite bottom plate. The bottom plate provides an oscillating shear stress to the fluid. Due to this shear stress, the fluid starts motion in its plane. Exact solutions of velocity and shear stress are determined for the described motion by using the integral transforms, namely Fourier cosine and sine transforms. These solutions are expressed as a combination of transient and steady-state solutions and they satisfy the governing equation as well as all the implemented initial and boundary conditions. The obtained solutions describe the flow of Casson fluid after some time of its initiation. The transient part of the solution vanishes after that initiation time and we get the steady-state solutions. In addition, these solutions are free from initial conditions and periodic in time. To analyze the effect of side walls on the flow of Casson fluid and also to search out some of the related physical aspects of the calculated results, the figures for velocities $u_s(y, t)$ and $u_c(y, t)$ over an infinite

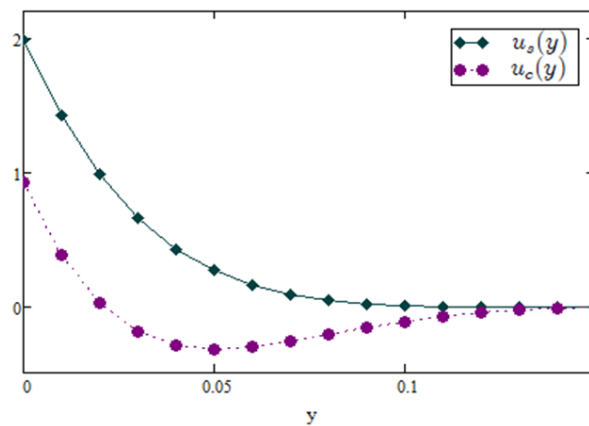


Figure 2: The comparison of velocities and $u_c(y)$, given by Eqs. (16) and (20), for $f = -5, \nu = 0.0102, \omega = 2, z = 0, t = 1s, \mu = 1.002$, and $\beta = 10$

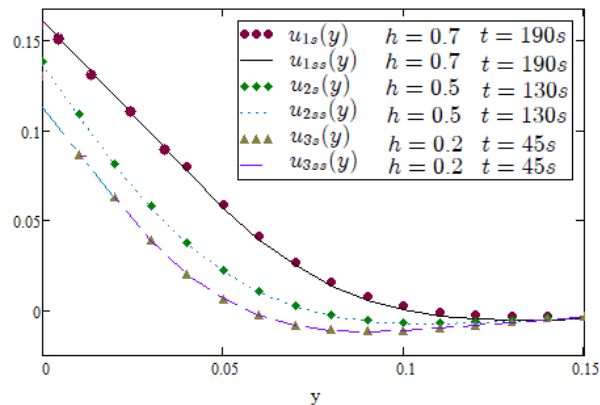


Figure 3: Eqs. (16) and (18) present the profile of velocities $u_s(y)$ and $u_{ss}(y)$, for various values of h and $f = -5, \nu = 0.0102, \omega = 2, z = 0, \mu = 1.002, \beta = 2$

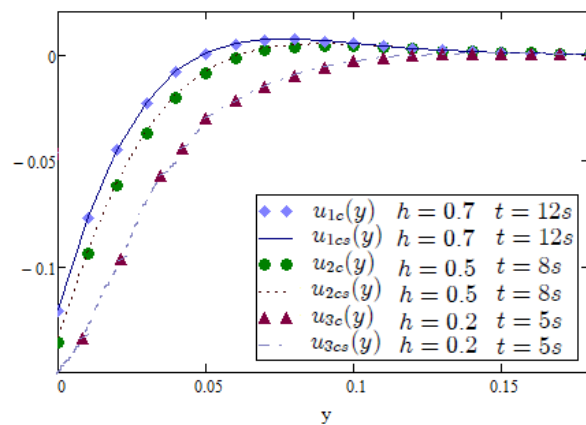


Figure 4: Eqs. (20) and (22) present the profile of velocities $u_c(y)$ and $u_{cs}(y)$, for different values and $f = -5, \nu = 0.0102, \omega = 2, z = 0, \mu = 1.002, \beta = 2$

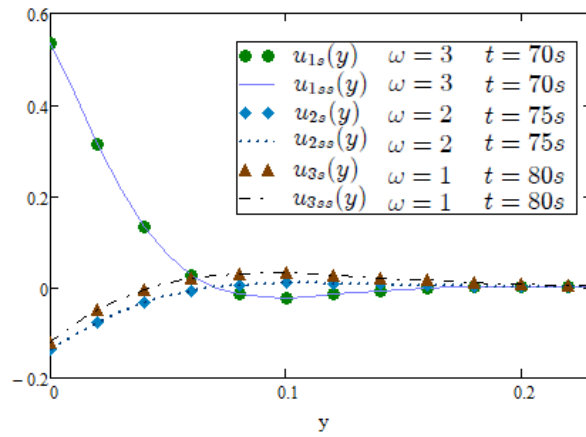


Figure 5: Eqs. (16) and (18) present the profiles of velocities $u_s(y)$ and $u_{ss}(y)$, for different values ω and $f = -5, \nu = 0.0102, h = 0.2, z = 0, \mu = 1.002, \beta = 2$

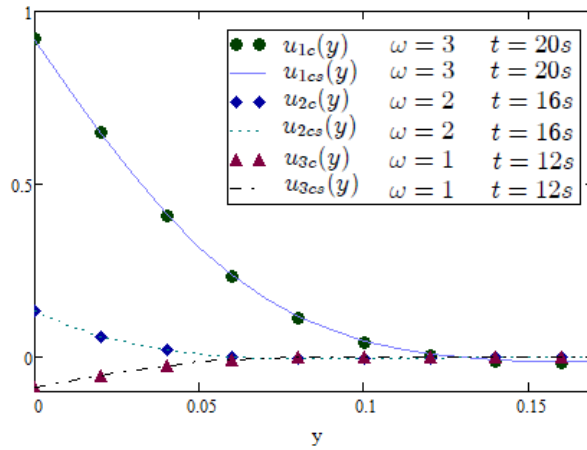


Figure 6: Eqs. (20) and (22) present the profile of velocities $u_c(y)$ and $u_{cs}(y)$, for different values ω and $f = -5, \nu = 0.0102, h = 0.2, z = 0, \mu = 1.002, \beta = 2$

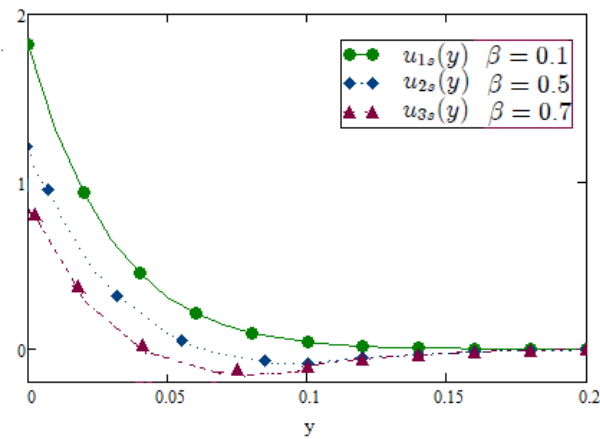


Figure 7: Eq. (16) presents the velocity profile $u_s(y)$, for various values of β and $f = -5, \nu = 0.0102, t = 3.45, z = 0, \mu = 1.02$

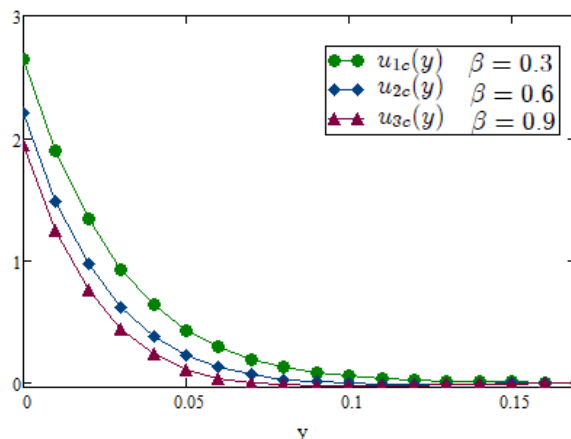


Figure 8: Eq. (20) presents the velocity profile $u_c(y)$, for various values of β and $f = -5$, $\nu = 0.0102$, $t = 2.9$, $z = 0$, $\mu = 1.002$

plate as well as in the middle of the channel have been plotted for some values of t and material constants against y . In Fig. 2 we note that when the time approaches to zero, the velocity field $u_s(y, t)$ for sine oscillation increase due to the absence of shear stress and the velocity field for cosine oscillation reduce due to some constant shear stress greater than zero. In practice, it is also necessary to know about the time which is required for attaining the steady state. The required time has been found in Figs. 3 and 4 for various values of h . As expected, the time required to attain the steady state increases if the distance between the side walls increases. In addition, this time value for sine oscillation is greater than cosine oscillation of the shear stress on the boundary. In Figs. 5 and 6, we noted that the time required to reach the steady state increases if the frequency ω of the shear stress decreases. The influence of the Casson fluid parameter on velocity profiles is shown in Figs. 7 and 8. It is found that there is an inverse variation between the Casson parameter β and velocity. It is pertinent to mention that an increase in the numerical value of Casson parameter decreases the thickness of the velocity boundary layer. It has been noticed that for extremely large values of β , i.e. $\beta \rightarrow \infty$, the non-Newtonian Casson fluid takes the behavior of Newtonian fluid. The units of material constants in Figs. 2 to 8 are SI units.

6 Conclusion

In this paper, exact solutions for velocity and shear stress of the unsteady unidirectional flow of Casson fluid between two side parallel side walls are examined. The plate applies an oscillating shear stress to the fluid. The general form of the exact solutions is determined with the help of integral transforms. The obtained solution satisfies all the imposed initial and boundary conditions. The Newtonian solutions are determined as limiting cases of the general solutions. Furthermore, they can also be used to give the solutions owing to the flow of fluid over an infinite plate that exerts the same shear stresses to the fluid, and various known solutions from the literature are obtained as limiting cases of our solutions. The results are plotted and it is found that velocity is the decreasing function of the Casson parameter. It is also noted that, when the distance between the side walls increases, the time required to reach the steady state also increases.

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Conflicts of Interest The author declares no conflict of interest.

References

- [1] W. Chaoyang and T. Chuanjing, "Boundary-layer flow and heat transfer of non-Newtonian fluids in porous media," *Int. J. Heat Fluid Flow*, vol. 10, no. 2, pp. 160–165, 1989.
- [2] M. Qasim, "Heat and mass transfer in a Jeffrey fluid over a stretching sheet with heat source/sink," *Alexandria Eng. J.*, vol. 52, no. 4, pp. 571–575, 2013.

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- [3] T. Hayat and Z. Abbas, "Heat transfer analysis on the MHD flow of a second grade fluid in a channel with porous medium," *Chaos, Solitons Fractals*, vol. 38, no. 2, pp. 556–567, 2008.
- [4] O. Z. Sharaf, R. A. Taylor, and E. Abu-Nada, "On the colloidal and chemical stability of solar nanofluids: From nanoscale interactions to recent advances," *Phys. Rep.*, vol. 867, pp. 1–84, 2020.
- [5] M. A. Hassan, M. Pathak, and M. K. Khan, "Natural convection of viscoplastic fluids in a square enclosure," *J. Heat Transfer*, vol. 135, no. 12, p. 122501, 2013.
- [6] J. Kleppe and W. J. Marner, "Transient free convection in a Bingham plastic on a vertical flat plate," 1972.
- [7] M. N. Zakaria, A. Hussanan, I. Khan, and S. Shafie, "The effects of radiation on free convection flow with ramped wall temperature in Brinkman type fluid," *J. Teknol.*, vol. 62, no. 3, pp. 33–39, 2013.
- [8] V. R. Prasad, B. Vasu, B. O. Anwar, and R. Parshad, "Unsteady free convection heat and mass transfer in a Walters-B viscoelastic flow past a semi-infinite vertical plate: a numerical study," *Therm. Sci.*, vol. 15, no. suppl. 2, pp. 291–305, 2011.
- [9] I. Khan, K. Fakhar, and M. I. Anwar, "Hydromagnetic rotating flows of an Oldroyd-B fluid in a porous medium," *Spec. Top. Rev. Porous Media An Int. J.*, vol. 3, no. 1, 2012.
- [10] N. Casson, "Flow equation for pigment-oil suspensions of the printing ink-type," *Rheol. Disperse Syst.*, pp. 84–104, 1959.
- [11] H. R. Kataria and H. R. Patel, "Soret and heat generation effects on MHD Casson fluid flow past an oscillating vertical plate embedded through porous medium," *Alexandria Eng. J.*, vol. 55, no. 3, pp. 2125–2137, 2016.
- [12] A. Subba Rao, V. Ramachandra Prasad, N. Bhaskar Reddy, and O. Anwar Bég, "Heat transfer in a Casson rheological fluid from a semi-infinite vertical plate with partial slip," *Heat Transf. Res.*, vol. 44, no. 3, pp. 272–291, 2015.
- [13] M. Mustafa, T. Hayat, I. Pop, and A. Aziz, "Unsteady boundary layer flow of a Casson fluid due to an impulsively started moving flat plate," *Heat Transf. Res.*, vol. 40, no. 6, pp. 563–576, 2011.
- [14] A. Hussanan, M. Zuki Salleh, R. M. Tahar, and I. Khan, "Unsteady boundary layer flow and heat transfer of a Casson fluid past an oscillating vertical plate with Newtonian heating," *PLoS One*, vol. 9, no. 10, p. e108763, 2014.
- [15] C. S. K. Raju, N. Sandeep, and S. Saleem, "Effects of induced magnetic field and homogeneous–heterogeneous reactions on stagnation flow of a Casson fluid," *Eng. Sci. Technol. an Int. J.*, vol. 19, no. 2, pp. 875–887, 2016.
- [16] K. Bhattacharyya, "Boundary layer stagnation-point flow of Casson fluid and heat transfer towards a shrinking/stretching sheet," *Front. Heat Mass Transf.*, vol. 4, no. 2, 2013.
- [17] K. Bhattacharyya, T. Hayat, and A. Alsaedi, "Exact solution for boundary layer flow of Casson fluid over a permeable stretching/shrinking sheet," *ZAMM-Journal Appl. Math. Mech. für Angew. Math. und Mech.*, vol. 94, no. 6, pp. 522–528, 2014.
- [18] A. Khalid, I. Khan, A. Khan, and S. Shafie, "Unsteady MHD free convection flow of Casson fluid past over an oscillating vertical plate embedded in a porous medium," *Eng. Sci. Technol. an Int. J.*, vol. 18, no. 3, pp. 309–317, 2015.
- [19] G. Makanda, S. Shaw, and P. Sibanda, "Effects of radiation on MHD free convection of a Casson fluid from a horizontal circular cylinder with partial slip in non-Darcy porous medium with viscous dissipation," *Bound. Value Probl.*, vol. 2015, no. 1, pp. 1–14, 2015.
- [20] G. Makanda, S. Shaw, and P. Sibanda, "Diffusion of chemically reactive species in Casson fluid flow over an unsteady stretching surface in porous medium in the presence of a magnetic field," *Math. Probl. Eng.*, vol. 2015, 2015.

- [21] D. Vieru, C. Fetecau, and C. Fetecau, "Flow of a viscoelastic fluid with the fractional Maxwell model between two side walls perpendicular to a plate," *Appl. Math. Comput.*, vol. 200, no. 1, pp. 459–464, 2008.
- [22] C. Fetecau, C. Fetecau, M. Kamran, and D. Vieru, "Exact solutions for the flow of a generalized Oldroyd-B fluid induced by a constantly accelerating plate between two side walls perpendicular to the plate," *J. Nonnewton. Fluid Mech.*, vol. 156, no. 3, pp. 189–201, 2009.
- [23] M. E. Erdoğan and C. E. İmrak, "Some effects of side walls on unsteady flow of a viscous fluid over a plane wall," *Math. Probl. Eng.*, vol. 2009, 2009.
- [24] C. Fetecau, M. Rana, and C. Fetecau, "Radiative and porous effects on free convection flow near a vertical plate that applies shear stress to the fluid," *Zeitschrift für Naturforsch. A*, vol. 68, no. 1–2, pp. 130–138, 2013.
- [25] Y. Yao and Y. Liu, "Some unsteady flows of a second grade fluid over a plane wall," *Nonlinear Anal. Real World Appl.*, vol. 11, no. 5, pp. 4442–4450, 2010.
- [26] D. Vieru, C. Fetecau, and A. Sohail, "Flow due to a plate that applies an accelerated shear to a second grade fluid between two parallel walls perpendicular to the plate," *Zeitschrift für Angew. Math. und Phys.*, vol. 62, pp. 161–172, 2011.
- [27] C. Fetecau, C. Fetecau, and M. Rana, "General solutions for the unsteady flow of second-grade fluids over an infinite plate that applies arbitrary shear to the fluid," *Zeitschrift für Naturforsch. A*, vol. 66, no. 12, pp. 753–759, 2011.
- [28] D. Vieru, C. Fetecau, and M. Rana, "Starting solutions for the flow of second grade fluids in a rectangular channel due to an oscillating shear stress," in *AIP Conference Proceedings*, 2012, vol. 1450, no. 1, pp. 45–54.
- [29] N. Shahid, M. Rana, and M. A. Imran, "Flow of an oldroyd-B fluid over an infinite plate subject to a time-dependent shear stress," *J. Prime Res. Math.*, vol. 7, pp. 52–62, 2011.
- [30] A. Sohail, D. Vieru, and M. A. Imran, "Influence of side walls on the oscillating motion of a Maxwell fluid over an infinite plate," *Mechanics*, vol. 19, no. 3, pp. 269–276, 2013.
- [31] S. Nadeem, R. U. Haq, and C. Lee, "MHD flow of a Casson fluid over an exponentially shrinking sheet," *Sci. Iran.*, vol. 19, no. 6, pp. 1550–1553, 2012.
- [32] I. N. Sneddon and F. Transforms, "McGraw Hill Book Co," Inc., New York, 1951.
- [33] I. N. Sneddon, "Functional analysis,[in:] *Encyclopedia of Physics*, S. Flügge [Ed.], Vol. II." Springer, Berlin-Göttingen-Heidelberg, 1955.
- [34] C. Fetecau, D. Vieru, and C. Fetecau, "Effect of side walls on the motion of a viscous fluid induced by an infinite plate that applies an oscillating shear stress to the fluid," *Open Phys.*, vol. 9, no. 3, pp. 816–824, 2011.