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The dynamics of the Zika with optimal Control strategies

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A B S T R A C T

We proposed a mathematical model on Zika virus and presented its global dynamics with optimal control strategies. The basic model formulation and its mathematical results are presented. The proposed Zika model is locally asymptotically stable whenever the basic reproduction number $\mathcal{R}_0<1$ (disease free case) and $\mathcal{R}_0>1$ (endemic case). We show mathematical results for the global stability of the Zika model. The Zika model is globally asymptotically stable for the case of disease free when $\mathcal{R}_0<1$ and whenever $\mathcal{R}_0>1$, the model is globally asymptotically stable at the endemic state. We present an optimal control model for the dynamics of Zika virus with three controls, (the minimization of contacts among humans and mosquitoes by wearing long sleeve shirts, big trousers, stay in places with screen window to keep the mosquito outside, sleep under bed net), (the contacts from mosquitoes to humans individuals by increasing the auto immunity), (increasing the death rate of mosquitos by using the insecticide spraying). The numerical simulation is performed for both the systems and the corresponding results are presented in graphical shape with different strategies. Finally, the brief conclusion is presented with source of references.

1. Introduction

Zika virus is a kind of virus which spreads by the bit of an effected Aedes mosquito. It has first discovered in Uganda in 1947. Since the identification of the first case in 2007, the virus quickly spreads to other parts of the world such as Africa, America and Asia [1]. These viruses can be transmitted through folk frequent moment from one place to another and with social interaction. These two factor aggravate the problem and some recent cases were reported in People's Republic of China. Female Aedes mosquitoes is the main source of Zika infection in human community with the same like the other vector borne disease such as Dengue, yellow fever and chikungunya [2]. People with Zika will get mild symptom and will get

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mild illness, they get severe Ailment. The Zika effected people usually do not get enough sick and they rarely die of this disease. Zika effected patient might be having different symptoms such as kin rashes, headache, conjunctivitis and muscle pains and these symptoms can last for many days to week. Some other information about the Zika disease can be found in [1]. Zika can also be transmitted between two couple having unprotected intercourse , if one partner is effected from Zika virus. It can be passed through effected pregnant women to her developing fetus [3]-[4]. If this happened the defect will likely to occur in newly born babies with abnormal brain and small head development, also muscle weakness caused by immune system which effect nervous system which also reported Zika in Brazil[5]. The information about the pandemic threats and other risk are reported in [6].

Some mathematical models regarding the Zika virus epidemics and outbreaks are available in literature. For the 2013-2014 outbreak of Zika virus through a mathematical modeling is proposed in [7]. The Zika model with in the scope of fractional derivative is considered in [9] and some other study about the dynamics of Zika through a mathematical model can be seen [10]. Some mathematical models that highlight the issue of Zika complications and other related issues are discussed [9]-[10]. A mathematical model with optimal control analysis is proposed in [11]. A theoretical model that investigated the Zika dynamics with control analysis is presented in [15]. Some recent work regarding the dynamics of Zika virus with different aspects have been studied in [12, 13, 14]. A few studied presented above focused on the how to control this Zika infection in the community. Here, our focus is to investigate the dynamics of the Zika virus though the available control and give some useful control strategies for the disease elimination.

The paper is divided in different sections and as is follows: The dynamics of Zika and relevant issue with details of the available study is briefly discussed in section introduction. We give a brief analysis about the Zika model in section 2. The fundamental properties of the Zika model is presented in section 3. Section 4 is used to discussed the stability results for the Zika model. The control problem with controls and their formulation with detailed mathematical results are presented in section 5. Numerical results with brief discussion is given in section 6 while the results are summarized in section 7.

2 Mathematical Model Formulation

The model considered in [11] is only some control measure and we used the model to present some global and local results and then briefly provide the control strategies for the Zika infection elimination. Here in this model, we divide the human population into two sub-classes, susceptible individuals and infected individuals. The human population is represented by $N_h(t)$,where H_s represent susceptible and H_i represent numbers of infected individuals.Thus,

$$N_h(t) = H_s(t) + H_i(t)$$

Similar , by N_m we represent the total number of mosquitos which are divide into susceptible M_s, and infected mosquitos M_i. So that

$$N_m(t) = M_s(t) + M_i(t)$$

The compartmentally mathematical model is given by following system of four ordinary differential equations to describe the mechanism of the transmission of Zika virus.

$$\begin{cases} \frac{dH_s}{dt} = \Lambda_h - \beta_{mh}\gamma_{mh}H_s(t)M_i(t) - d_hH_s(t), \\ \frac{dH_i}{dt} = \beta_{mh}\gamma_{mh}H_s(t)M_i(t) - d_hH_i(t), \\ \frac{dM_s}{dt} = \Lambda_m - \beta_{hm}\gamma_{hm}H_i(t)M_s(t) - d_mM_s(t), \\ \frac{dM_i}{dt} = \beta_{hm}\gamma_{hm}H_i(t)M_s(t) - d_mM_i(t), \end{cases} \tag{1}$$

with the given initial conditions

$$H_s(0) = H_{s0} \geq 0,\ H_i(0) = H_{i0} \geq 0,\ M_s(0) = M_{s0} \geq 0,\ M_i(0) = M_{i0} \geq 0.$$

In model (1), Λ_h is recruitment rate of H_s, the recruitment rate of susceptible mosquitos is Λ_m , d_m is the mosquitos natural death rate while and d_h represent the natural death rate of humans. The contact rate among human and misquotes is given by β_{mh} while from infected mosquitos to human individuals the contact rate is given by γ_{mh}. Obviously, $\beta_{mh} = \beta_{hm}$, but the rate of transmission is different from M_i-to-H_s and H_i-to-M_s.

3 Mathematical Analysis

We provide some mathematical results for the Zika model (1) in the following.

Theorem 3.1. *For the provided set of initial values for the system (1) which are* $H_s(0) > 0$,$H_i(0) > 0$, $M_s(0) > 0$ *and* $M_i(0) > 0$. *Then we have* $(H_s(t), H_i(t), M_s(t), M_i(t)) : (0,\infty) \longrightarrow (0,\infty)$ *which provide solution to the model (1).*

Proof. Consider that

$$t^* = sup\{t > 0, H_s > 0, H_i > 0, M_s > 0, M_i > 0\} \in [0, t]. \tag{2}$$

So, $t^* > 0$, and using the first equation of the Zika model (1), we have the result,

$$\frac{dH_s}{dt} = \Lambda_h - (\lambda + d_h)H_s \quad where \quad \lambda = (\beta_{mh}\gamma_{mh}M_i).$$

So, we have

$$\frac{d}{dt}\Big[H_s(t)\ exp\Big\{d_ht + \int_0^t \lambda(s)ds\Big\}\Big] = \Lambda_h\ exp\Big[d_ht + \int_0^t \lambda(s)ds\Big].$$

So,

$$S(t^*)\ exp\Big[d_ht^* + \int_0^{t^*} \lambda(s)ds\Big] - H_s(0) = \int_0^{t^*} \Lambda_h\ exp\Big[d_ht^* + \int_0^t \lambda(v)dv\Big]dt^*,$$

giving

$$S(t^*) = H_s(0)exp\Big[-\Big(d_ht^* + \int_0^{t^*} \lambda(s)ds\Big)\Big] + exp\Big[-\Big(d_ht^* + \int_0^{t^*} \lambda(s)ds\Big)\Big]$$

$$\times\Big[\int_0^{t^*}\Lambda_h\ exp\Big[d_h t^* + \int_0^t \lambda(v)dv\Big]dt^*\Big] > 0.$$

Using the second equation of the Zika model (1), the following is presented,

$$\frac{dH_i}{dt} = \lambda H_s - d_h H_i \geq -d_h H_i,$$

which implies

$$H_i(t^*) \geq H_{i(0)} e^{-d_h t^*} > 0.$$

We have the results similarly for the rest of the equations and obtain $M_s(t) > 0$, and $M_i(t) > 0$. □

3.1 Invariant region

From system (1), it can be written as

$$\frac{dN_h}{dt} = \Lambda_h - d_h N_h$$

i.e.,

$$\frac{dN_h}{dt} + d_h N_h \leq \Lambda_h.$$

integrating both sides with the application theory of differential equations [18], we have

$$0 \leq N_h\Big(H_s, H_i\Big) \leq \frac{\Lambda_h}{d_h}(1 - e^{-d_h t}) + N_h(H_s(0) + H_i(0))e^{-d_h t}.$$

Taking, $t \to \infty$, we obtain $0 < N_m \leq \frac{\Lambda_h}{d_h}$. The total mosquitos population is

$$\frac{dN_m}{dt} = \Lambda_m - d_m N_m. \tag{3}$$

Using the same procedure that is for human population, we obtain

$$0 \leq N_m\Big(M_s, M_i\Big) \leq \frac{\Lambda_m}{d_m}(1 - e^{-d_m t}) + N_m(M_s(0) + M_i(0))e^{-d_m t}.$$

Taking, $t \to \infty$, we obtain $0 < N_m \leq \frac{\Lambda_m}{d_m}$ So, the biological feasible region for (1) is

$$\Phi = \Big\{(H_s, H_i, M_s, M_i) \in \mathbb{R}^4_+ | 0 \leq H_s + H_i \leq \frac{\Lambda_h}{d_h},\ 0 \leq M_s + M_i \leq \frac{\Lambda_m}{d_m}\Big\},$$

where the existence, uniqueness, and continuity results holds. We can easily establish the positive invariance of Φ. It thus suffices to consider the dynamics of our system in Φ, where the model is meaningful both epidemiologically and mathematically.

4 Equilibria

The Zika model given by (1) has the infection free equilibrium denoted by $\mathcal{E}^0$, is given by

$$\mathcal{E}^0 = \left(H_s^0, H_i^0, M_s^0, M_i^0\right) = \left(\frac{\Lambda_h}{d_h}, 0, \frac{\Lambda_m}{d_m}, 0\right).$$

The computation of the basic reproduction for any epidemiological model provide the information of the disease progress in the community. It is simply explained by entering an average number of infection to populations that is purely susceptible and produced other secondary infections. For the given Zika model (1), we compute the basic reproduction number by using the technique presented in [19]. The relevant results of the computation of the matrices F and V are shown below

$$F = \begin{pmatrix} 0 & \beta_{mh}\gamma_{mh}\frac{\Lambda_h}{d_h} \\ \beta_{hm}\gamma_{hm}\frac{\Lambda_m}{d_m} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} d_h & 0 \\ 0 & d_m \end{pmatrix},$$

$$V^{-1} = \begin{pmatrix} \frac{1}{d_h} & 0 \\ 0 & \frac{1}{d_m} \end{pmatrix}, \; FV^{-1} = \begin{pmatrix} 0 & \frac{\beta_{mh}\gamma_{mh}\Lambda_h}{d_h d_m} \\ \frac{\beta_{hm}\gamma_{hm}\Lambda_m}{d_h d_m} & 0 \end{pmatrix}.$$

We have

$$FV^{-1} = \begin{pmatrix} 0 & \frac{\beta_{mh}\gamma_{mh}\Lambda_h}{d_h d_m} \\ \frac{\beta_{hm}\gamma_{hm}\Lambda_m}{d_h d_m} & 0 \end{pmatrix}.$$

The eigenvalues associated to FV^{-1} are $\left\{-\frac{\sqrt{\Lambda_h\beta_{hm}\gamma_{hm}\Lambda_m\beta_{mh}\gamma_{mh}}}{d_h d_m}, \frac{\sqrt{\Lambda_h\beta_{hm}\gamma_{hm}\Lambda_m\beta_{mh}\gamma_{mh}}}{d_h d_m}\right\}$. The spectral radius $\rho(FV^{-1})$ is the basic reproduction number of the model and after some rigorous computations and simplification the reproduction number is

$$\mathcal{R}_0 = \sqrt{\frac{\Lambda_h\beta_{hm}\gamma_{hm}\Lambda_m\beta_{mh}\gamma_{mh}}{d_h^2 d_m^2}}.$$

Next section explores the stability analysis of the Zika model (1). So, we have the following results:

4.1 Local Stability

Theorem 4.1. *The Zika model given by (1) is locally asymptotically stable if* $\mathcal{R}_0 < 1$.

Proof. The jacobian matrix evaluated at $\mathcal{E}^0$ is

$$J(\mathcal{E}^0) = \begin{pmatrix} -d_h & 0 & 0 & -\frac{\beta_{mh}\gamma_{mh}\Lambda_h}{d_h} \\ 0 & -d_h & 0 & \frac{\beta_{mh}\gamma_{mh}\Lambda_h}{d_h} \\ 0 & -\frac{\beta_{hm}\gamma_{hm}\Lambda_m}{d_m} & -d_m & 0 \\ 0 & \frac{\beta_{hm}\gamma_{hm}\Lambda_m}{d_m} & 0 & -d_m \end{pmatrix}$$

two root of $J(\mathcal{E}^0)$ is $-d_h$ and $-d_m$ is clearly negative. The other roots are obtained through the following equations:

$$x^2 + x(d_h + d_m) + d_h d_m(1 - \mathcal{R}_0) = 0.$$

So, it is obvious from the quadratic equation that the Zika model given by (1) is locally asymptotically stable, whenever $\mathcal{R}_0 < 1$. □

4.2 Endemic Equilibria

The endemic equilibria of the model (1) denoted by $E_1 = \left(H_s^{**}, H_i^{**}, M_s^{**}, M_i^{**}\right)$ and is given by

$$\begin{aligned} H_s^{**} &= \frac{\Lambda_h}{d_h + M_i^{**}\beta_{mh}\gamma_{mh}}, \quad H_i^{**} = \frac{\Lambda_h M_i^{**}\beta_{mh}\gamma_{mh}}{d_h\left(d_h + M_i^{**}\beta_{mh}\gamma_{mh}\right)}, \\ M_s^{**} &= \frac{\Lambda_m\left(d_h M_i^{**}\beta_{mh}\gamma_{mh} + d_h^2\right)}{d_m\left(d_h M_i^{**}\beta_{mh}\gamma_{mh} + d_h^2\right) + \Lambda_h\beta_{hm}\gamma_{hm}M_i^{**}\beta_{mh}\gamma_{mh}} \end{aligned}$$

Based on the existence of the above endemic equilibria we have the following result:

Theorem 4.2. *The Zika model given by (1) at the endemic state is locally asymptotically stable if* $\mathcal{R}_0 > 1$.

Proof. The following Jacobian matrix is presented at the endemic state E_1,

$$J(E_1) = \begin{pmatrix} -Q_1 - d_h & 0 & 0 & -Q_3 \\ Q_1 & -d_h & 0 & Q_3 \\ 0 & -Q_4 & -Q_2 - d_m & 0 \\ 0 & Q_4 & Q_2 & -d_m \end{pmatrix} \tag{4}$$

The characteristics equation of $J(E_1)$ is

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0,$$

where

$$\begin{aligned} c_1 &= 2\left(d_h + d_m\right) + Q_1 + Q_2, \\ c_2 &= d_h\left(d_h + 4d_m + Q_1 + 2Q_2\right) + d_m\left(d_m + 2Q_1 + Q_2\right) + Q_1Q_2 - Q_3Q_4, \\ c_3 &= d_h\left(2d_m(Q_1 + Q_2) + 2d_m^2 + Q_1Q_2 - Q_3Q_4\right) + d_h^2\left(2d_m + Q_2\right) + d_m\left(Q_1d_m + Q_1Q_2 - Q_3Q_4\right), \\ c_4 &= d_h d_m\left(d_h\left(d_m + Q_2\right) + Q_1d_m + Q_1Q_2 - Q_3Q_4\right). \end{aligned} \tag{5}$$

Where $Q_1 = M_i^{**}\beta_{m_h}\gamma_{m_h}, Q_2 = H_i^{**}\beta_{h_m}\gamma_{h_m}, Q_3 = H_s^{**}\beta_{m_h}\gamma_{m_h}, Q_4 = M_s^{**}\beta_{h_m}\gamma_{h_m}$.

$$H_1 = c_1,\ H_2 = \begin{pmatrix} c_1 & 1 \\ c_3 & c_2 \end{pmatrix},\quad H_3 = \begin{pmatrix} c_1 & 1 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{pmatrix},\quad H_4 = \begin{pmatrix} c_1 & 1 & 0 & 0 \\ c_3 & c_2 & c_1 & 0 \\ 0 & c_4 & c_3 & c_2 \\ 0 & 0 & 0 & c_4 \end{pmatrix}. \tag{6}$$

It can be seen that coefficients $c_i > 0$, for $i = 1, 2, 3, 4$ is necessary and we have four eigenvalues with negative real parts. So, the Routh-Hurtwiz criteria ensures the local stability of the Zika model given by (1) at the given endemic state. □

4.3 Global stability DFE

This subsection provide the global analysis of the Zika model for the infection free and the endemic equilibrium. The following results are presented:

$$V\Big(H_s, H_i, M_s, M_i\Big) = w_1 \int_{H_s^0}^{H_s} (1 - \frac{H_s^0}{z})dz + w_2 H_i + w_3 \int_{M_s^0}^{M_s} (1 - \frac{M_s^0}{z})dz + w_4 M_i, \qquad (7)$$

where w_i for $i = 1, 2..., 4$ are the positive constants that will be chosen later while V is a lyapunove function. The differentiation of (7), with the use of model (1), we have

$$\begin{aligned}
V' &= w_1\Big(\tfrac{H_s - H_s^0}{H_s}\Big)H_s' + w_2 H_i' + w_3\Big(\tfrac{M_s - M_s^0}{M_s}\Big)M_s' + w_4 M_i', \\
&= w_1\Big(\tfrac{H_s - H_s^0}{H_s}\Big)\Big[\Lambda_h - d_h H_s - \beta_{mh}\gamma_{mh} M_i H_s\Big] + w_2\Big[\beta_{mh}\gamma_{mh} M_i H_s - d_h H_i\Big] \\
&\quad + w_3\Big(\tfrac{M_s - M_s^0}{M_s}\Big)\Big[\Lambda_m - d_m M_s - \beta_{hm}\gamma_{hm} M_s H_i\Big] + w_4\Big[\beta_{hm}\gamma_{hm} M_s H_i - d_m M_i\Big] \\
&= (w_2 - w_1)\Big[\beta_{mh}\gamma_{mh} M_i H_s\Big] + (w_4 - w_3)\Big[\beta_{hm}\gamma_{hm} M_s H_i\Big] \\
&\quad + M_i(w_1\beta_{mh}\gamma_{mh} H_s^0 - w_4 d_m) + H_i(w_3\beta_{hm}\gamma_{hm} M_s^0 - w_2 d_h)
\end{aligned}$$

Using $H_s^0 = \frac{\Lambda_h}{d_h}$ and $M_s^0 = \frac{\Lambda_m}{d_m}$, we get

$$\begin{aligned}
V' &= (w_2 - w_1)\Big[\beta_{mh}\gamma_{mh} M_i H_s\Big] + (w_4 - w_3)\Big[\beta_{hm}\gamma_{hm} M_s H_i\Big] \\
&\quad + M_i(w_1\beta_{mh}\gamma_{mh}\tfrac{\Lambda_h}{d_h} - w_4 d_m) + H_i(w_3\beta_{hm}\gamma_{hm}\tfrac{\Lambda_m}{d_m} - w_2 d_h)
\end{aligned}$$

Choosing the constants $w_1 = w_2 = \beta_{hm}\gamma_{hm}\frac{\Lambda_m}{d_m}$ and $w_3 = w_4 = d_h$ and after simplification, we get

$$V' = M_i d_h d_m\Big(\mathcal{R}_0 - 1\Big).$$

$V'(t)$ is negative for $\mathcal{R}_0 < 1$ and "0" if $M_i = 0$. therefore, the largest compact invariant set in $\Big\{(H_s, H_I, M_s, M_i) \in \Phi | V'(t) = 0\Big\}$, when $\mathcal{R}_0 < 1$, is the singleton set $\mathcal{E}^0$. Hence by Lasalle's Invariance Principle [8], $\mathcal{E}^0$ is globally asymptotically stable in Φ.

4.4 Global stability Endemic Case

Before we show that global stability for the Zika model given by (1), first we have at the steady state the model (1) at the given equilibrium E_1,

$$\begin{cases} \Lambda_h = \beta_{mh}\gamma_{mh}M_i^{**}H_s^{**} + d_hH_s^{**}, \\ d_hH_i^{**} = \beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}, \\ \Lambda_m = \beta_{hm}\gamma_{hm}M_s^{**}H_i^{**} + d_mM_s^{**}, \\ d_mM_i^{**} = \beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}. \end{cases} \tag{8}$$

Based on the above we present the following result.

Theorem 4.3. *The Zika model given by (1) is globally asymptotically stable If* $\mathcal{R}_0 > 1$.

Proof. We consider the following lyapunove function:

$$L(t) = \int_{H_s^{**}}^{H_s}(1-\frac{H_s^{**}}{x})dx + \int_{H_i^{**}}^{H_i}(1-\frac{H_i^{**}}{x})dx + \int_{M_s^{**}}^{M_s}(1-\frac{M_s^{**}}{x})dx + \int_{M_i^{**}}^{M_i}(1-\frac{M_i^{**}}{x})dx.$$

The differentiation of $L(t)$ with the use of (1), we have

$$\dot{L} = \Big(1-\frac{H_s^{**}}{H_s}\Big)H_s' + \Big(1-\frac{H_i^{**}}{H_i}\Big)H_i' + \Big(1-\frac{M_s^{**}}{M_s}\Big)M_s' + \Big(1-\frac{M_i^{**}}{M_i}\Big)M_i'.$$

By direct calculations, we have that:

$$\begin{aligned} (1-\frac{H_s^{**}}{H_s})\frac{dH_s(t)}{dt} &= (1-\frac{H_s^{**}}{H_s})\Big(\Lambda_h - d_hH_s - \beta_{mh}\gamma_{mh}M_iH_s\Big), \\ (1-\frac{H_i^{**}}{H_i})\frac{dH_i(t)}{dt} &= (1-\frac{H_i^{**}}{H_i})\Big(\beta_{mh}\gamma_{mh}M_iH_s - d_hH_i\Big), \\ (1-\frac{M_s^{**}}{M_s})\frac{dM_s(t)}{dt} &= (1-\frac{M_s^{**}}{M_s})\Big(\Lambda_m - d_mM_s - \beta_{hm}\gamma_{hm}M_sH_i\Big), \\ (1-\frac{M_i^{**}}{M_i})\frac{dM_i(t)}{dt} &= (1-\frac{M_i^{**}}{M_i})\Big(\beta_{hm}\gamma_{hm}M_sH_i - d_mM_i\Big). \end{aligned} \tag{9}$$

$$\begin{aligned} (1-\frac{H_s^{**}}{H_s})\frac{dH_s(t)}{dt} &= (1-\frac{H_s^{**}}{H_s})\Big(\Lambda_h - d_hH_s - \beta_{mh}\gamma_{mh}M_iH_s\Big), \\ &= (1-\frac{H_s^{**}}{H_s})\Big(d_hH_s^{**} + \beta_{mh}\gamma_{mh}M_i^{**}H_s^{**} - d_hH_s - \beta_{mh}\gamma_{mh}M_iH_s\Big), \\ &= d_hH_s^{**}(1-\frac{H_s^{**}}{H_s})(1-\frac{H_s}{H_s^{**}}) + (1-\frac{H_s^{**}}{H_s})\Big(\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**} - \beta_{mh}\gamma_{mh}M_iH_s\Big), \\ &= d_hH_s^{**}\Big(2-\frac{H_s^{**}}{H_s}-\frac{H_s}{H_s^{**}}\Big) + \beta_{mh}\gamma_{mh}M_i^{**}H_s^{**} - \beta_{mh}\gamma_{mh}M_iH_s \end{aligned}$$

$$-\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}\frac{H_s^{**}}{H_s}+\beta_{mh}\gamma_{mh}M_iH_s^{**}. \tag{10}$$

$$\begin{aligned}
(1-\frac{H_i^{**}}{H_i})\frac{dH_i(t)}{dt} &= (1-\frac{H_i^{**}}{H_i})\Big(\beta_{mh}\gamma_{mh}M_iH_s-d_hH_i\Big),\\
&= \beta_{mh}\gamma_{mh}M_iH_s-d_hH_i-\beta_{mh}\gamma_{mh}M_iH_s\frac{H_i^{**}}{H_i}+d_hH_i^{**},\\
&= \beta_{mh}\gamma_{mh}M_iH_s-\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}\frac{H_i}{H_i^{**}}-\beta_{mh}\gamma_{mh}M_iH_s\frac{H_i^{**}}{H_i}+\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}. \qquad (11)
\end{aligned}$$

$$\begin{aligned}
(1-\frac{M_s^{**}}{M_s})\frac{dM_s(t)}{dt} &= (1-\frac{M_s^{**}}{M_s})\Big(\Lambda_m-d_mM_s-\beta_{hm}\gamma_{hm}M_sH_i\Big),\\
&= (1-\frac{M_s^{**}}{M_s})\Big(d_mM_s^{**}+\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}-d_mM_s-\beta_{hm}\gamma_{hm}M_sH_i\Big),\\
&= d_mM_s^{**}(1-\frac{M_s^{**}}{M_s})(1-\frac{M_s}{M_s^{**}})+(1-\frac{M_s^{**}}{M_s})\Big(\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}-\beta_{hm}\gamma_{hm}M_sH_i\Big),\\
&= d_mM_s^{**}\Big(2-\frac{M_s^{**}}{M_s}-\frac{M_s}{M_s^{**}}\Big)+\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}-\beta_{hm}\gamma_{hm}M_sH_i\\
&\quad -\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}\frac{M_s^{**}}{M_s}+\beta_{hm}\gamma_{hm}H_iM_s^{**}. \qquad (12)
\end{aligned}$$

$$\begin{aligned}
(1-\frac{M_i^{**}}{M_i})\frac{dM_i(t)}{dt} &= (1-\frac{M_i^{**}}{M_i})\Big(\beta_{hm}\gamma_{hm}M_sH_i-d_mM_i\Big),\\
&= \beta_{hm}\gamma_{hm}M_sH_i-d_mM_i-\beta_{hm}\gamma_{hm}M_sH_i\frac{M_i^{**}}{M_i}+d_mM_i^{**},\\
&= \beta_{hm}\gamma_{hm}M_sH_i-\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}\frac{M_i}{M_i^{**}}-\beta_{hm}\gamma_{hm}M_sH_i\frac{M_i^{**}}{M_i}+\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}. \qquad (13)
\end{aligned}$$

It follows from (10-13)

$$\begin{aligned}
\dot{L} &= d_hH_s^{**}\Big(2-\frac{H_s^{**}}{H_s}-\frac{H_s}{H_s^{**}}\Big)+\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}\Big(2-\frac{H_s^{**}}{H_s}-\frac{H_i}{H_i^{**}}-\frac{M_i}{M_i^{**}}\Big(\frac{H_sH_i^{**}}{H_s^{**}H_i}-1\Big)\Big)\\
&\quad +d_mM_s^{**}\Big(2-\frac{M_s^{**}}{M_s}-\frac{M_s}{M_s^{**}}\Big)+\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}\Big(2-\frac{M_s^{**}}{M_s}-\frac{M_i}{M_i^{**}}-\frac{H_i}{H_i^{**}}\Big(\frac{M_sM_i^{**}}{M_s^{**}M_i}-1\Big)\Big). \qquad (14)
\end{aligned}$$

In equation (14)

$$\dot{L}=d_hH_s^{**}\Big(2-\frac{H_s^{**}}{H_s}-\frac{H_s}{H_s^{**}}\Big)\le 0,$$

$$d_mM_s^{**}\Big(2-\frac{M_s^{**}}{M_s}-\frac{M_s}{M_s^{**}}\Big)\le 0,$$

$$\beta_{mh}\gamma_{mh}M_i^{**}H_s^{**}\Big(2-\frac{H_s^{**}}{H_s}-\frac{H_i}{H_i^{**}}-\frac{M_i}{M_i^{**}}\Big(\frac{H_sH_i^{**}}{H_s^{**}H_i}-1\Big)\Big)\leq,$$

$$\beta_{hm}\gamma_{hm}M_s^{**}H_i^{**}\Big(2-\frac{M_s^{**}}{M_s}-\frac{M_i}{M_i^{**}}-\frac{H_i}{H_i^{**}}\Big(\frac{M_sM_i^{**}}{M_s^{**}M_i}-1\Big)\Big)\leq 0.$$

Thus, the largest invariant subset, $\dot{L}(t)=0$ is E_1. So it follows from LaSalles invariance Principle [8], E_1 is globally asymptotically stable whenever $\mathcal{R}_0>1$. □

5 Application of optimal control theory to Zika virus

For the optimal control model formulation we use the three available controls which are defined briefly in the following:
• The control measure $u_1(t)$: represents the prevention from the Zika mosquitos by using full clothes, window screens, mosquito nets etc.
• $u_2(t)$: measure the minimization of the contact o of the transmission from infected mosquito to healthy people.
• $u_3(t)$: Using the spray to increase the death rate of mosquitos.
Using these controls in mind we formulate the optimal control problem in the following:

$$\begin{cases}\frac{dH_s}{dt}=\Lambda_h-(\beta_{mh}-u_1)(\gamma_{mh}-u_2)H_s(t)M_i(t)-d_hH_s(t),\\ \frac{dH_i}{dt}=(\beta_{mh}-u_2)(\gamma_{mh}-u_2)H_s(t)M_i(t)-d_hH_i(t),\\ \frac{dM_s}{dt}=\Lambda_m-\beta_{hm}\gamma_{hm}H_i(t)M_s(t)-(d_m+u_3)M_s(t),\\ \frac{dM_i}{dt}=\beta_{hm}\gamma_{hm}H_i(t)M_s(t)-(d_m+u_3)M_i(t),\end{cases}\tag{15}$$

with non-negative initial conditions.
The use of the three controls in the Zika control model (15), we have $u(t)=(u_1,u_2,u_3)\in\mathcal{S}$. These controls are taken in connection with the model variables and their control set is defined by $u(t)=(u_1,u_2,u_3)\in\mathcal{S}$ with H_s,H_i,M_s and M_i are bounded and

$$\mathcal{S}=\{(u_i,i=1,2,3)|u_i\ is\ Lebsegue\ measurable\ on\ [0,1],\ 0\leq u_i(t)\leq 1,\ t\in[0,T],\ i=1,2,3\}.\tag{16}$$

We have the objective equation for the Zika control (15),

$$J(u_1,u_2,u_3)=\int_0^{t_f}\Big(W_1H_i+W_2M_i+\frac{a_1}{2}u_1^2+\frac{a_2}{2}u_2^2+\frac{a_3}{2}u_3^2\Big)dt,\tag{17}$$

where W_1,W_2 respectively measure the preventions of the infected individuals, and the possible control strategies for the mosquitos elimination, while the other weight constants, a_i for $i=1,2,3$ is cost associated to the three controls, all these constants are positive. So, we have a control u_1^*,u_2^* and u_3^* such that,

$$J(u_1^*,u_2^*,u_3^*)=\min J(u_1,u_2,u_3).$$

The Hamiltonian for the control problem is given by,

$$\begin{aligned} H &= W_1 H_i(t) + W_2 M_i(t) + \tfrac{1}{2}\Big\{a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2\Big\} \\ &+\lambda_1 \{\Lambda_h - (\beta_{mh} - u_1)(\gamma_{mh} - u_2) H_s(t) M_i(t) - d_h H_s(t)\} \\ &+\lambda_2 \{(\beta_{mh} - u_1)(\gamma_{mh} - u_2) H_s(t) M_i(t) - d_h H_i(t)\} \\ &+\lambda_3 \{\Lambda_m - \beta_{hm}\gamma_{hm} H_i(t) M_s(t) - (d_m + u_3) M_s(t)\} \\ &+\lambda_4 \{\beta_{hm}\gamma_{hm} H_i(t) M_s(t) - (d_m + u_3) M_i(t)\} \end{aligned} \tag{18}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 represent the adjoint variables.

Theorem 5.1. *Given optimal controls u_1^*, u_2^*, u_3^* and solutions H_s, H_i, M_s, M_i of the corresponding state system (17)-(15) that minimize $J(u_1, u_2, u_3)$ over $\mathcal{S}$.Then there exists adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying*

$$\frac{-d\lambda_i}{dt} = \frac{\partial H}{\partial i} \tag{19}$$

where $i = \lambda_{H_s}, \lambda_{H_i}, \lambda_{M_s}, \lambda_{M_i}$ and with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0 \tag{20}$$

$$\begin{aligned} u_1^* &= \max\{\min\{1, \frac{(\lambda_2 - \lambda_1)(\gamma_{mh} - u_2) M_i(t) H_s(t)}{a_1}\}, 0\}, \\ u_2^* &= \max\{\min\{1, \frac{(\lambda_2 - \lambda_1)(\beta_{mh} - u_1) M_i(t) H_s(t)}{a_2}\}, 0\}, \\ u_3^* &= \max\{\min\{\frac{(\lambda_3 M_s + \lambda_4 M_i)}{a_3}\}, 0\}. \end{aligned} \tag{21}$$

Proof. The control problem presented above satisfy the conditions presented in [20] and so we obtained the following results:

$$\begin{cases} \lambda_1' = (\lambda_1 - \lambda_2)(\beta_{mh} - u_1)(\gamma_{mh} - u_2) M_i(t) + \lambda_1 d_h, \\ \lambda_2' = -W_1 + (\lambda_3 - \lambda_4)\beta_{hm}\gamma_{hm} M_s(t) + \lambda_2 d_h, \\ \lambda_3' = (\lambda_3 - \lambda_4)\beta_{hm}\gamma_{hm} H_i(t) + \lambda_3(d_m + u_3), \\ \lambda_4' = -W_2 + (\lambda_1 - \lambda_2)(\beta_{mh} - u_1)(\gamma_{mh} - u_2) H_s(t) + \lambda_4(d_m + u_3). \end{cases} \tag{22}$$

□

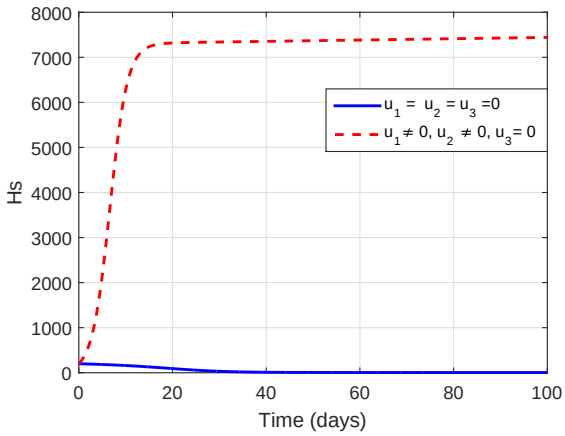
6 Numerical Results

In this section, we present the numerical solution of the optimality system in comparison with system(1). In graphical results the boldline shows the population without control while the dashed shows control system. The base line for the state variables in optimal

control system are used $H_s = 200$, $H_i = 10$,$M_s = 200$, $M_i = 20$. The wight and balancing constants used in numerical simulation except, $W_1 = 110$, $W_2 = 10$, $a_1 = 0.0050$, $a_2 = 0.30$ and $a_3 = 0.001$ are presented in Table 1. The numerical results are performed in matlab by using the RK-4, backward scheme. The time interval is taken days and belong to [0,100]. We presented the numerical results by using different set of controls. In Figure 1, we set $u_3 = 0$ and $u_1 = u_2 \neq 0$. In this set of control combination, the infected human, infected mosquitos decreases and the number of susceptible human and susceptible vector increases. In the second combination we set $u_2 = 0$ and $u_1 = u_3 \neq 0$. The corresponding graphical results for this combination is presented in Figure 2. In graphical results, the number of infected human, infected mosquitos and susceptible mosquito decreases while increase the population of susceptible human. In third set of combination (see figure 3), we set $u_1 = 0$ and $u - 2 = u_3 \neq 0$. In this set of control combination, the number of infected human, infected mosquitos and susceptible vector increases while increases the population of susceptible humans. Finally, the last set of control combination with all the controls are keep non-zero, and the corresponding graphical results are given in Figure 4. Here, we see that the number of infected human, infected mosquitos and susceptible mosquitos decreases while susceptible human increases high compare to other set of control and combinations. Thus, the combination four is the best for disease elimination in population by activating all the three controls at the same time.

Parameter	Description	value	Ref
Λ_h	Recruitment rate of human population	100 day^{-1}	Assumed
Λ_m	Recruitment rate of mosquitos population	1000 day^{-1}	Assumed
d_h	Natural death rate of human population	$1/(365 \times 60)$ day^{-1}	[22]
d_m	Natural death rate of mosquitos population	0.07 day^{-1}	assumed
β_{mh}	Contact rate between infected mosquitos and susceptible human	$0.2day^{-1}$	[21]
β_{mh}	Contact rate between susceptible mosquitos and infected human	0.09 day^{-1}	[22]
γ_{mh}	Transmission from infected mosquitos to susceptible human	0.11 day^{-1}	Assume
γ_{hm}	Transmission from infected human to susceptible mosquitos	0.122 day^{-1}	Assumed

Table 1: Variables and parameters description used in optimal control solution.



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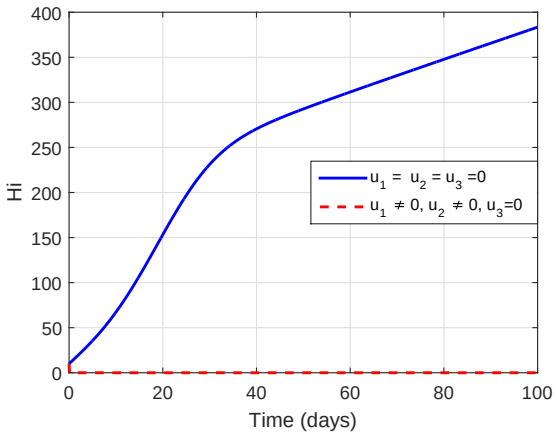

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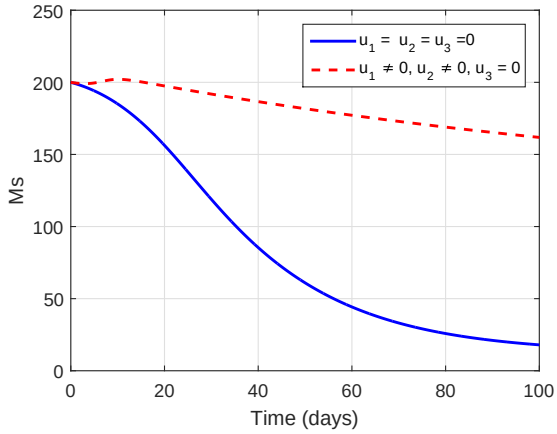

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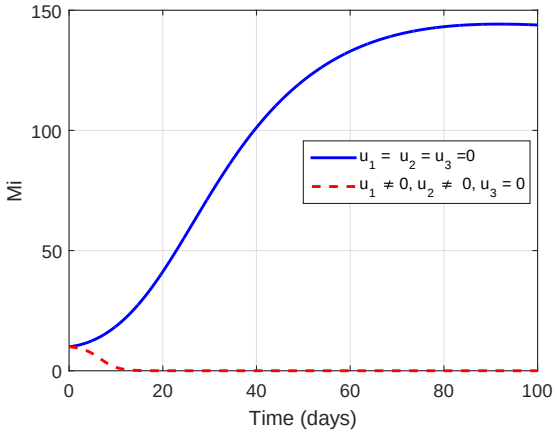

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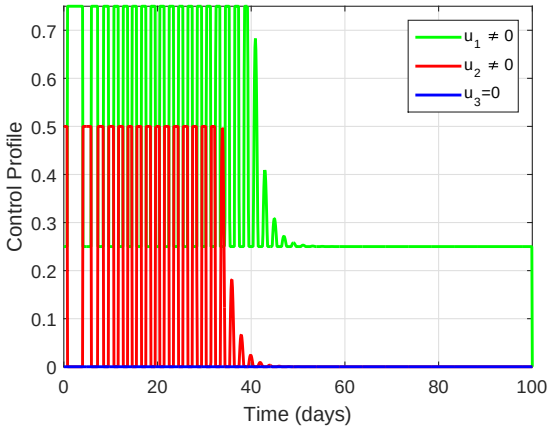
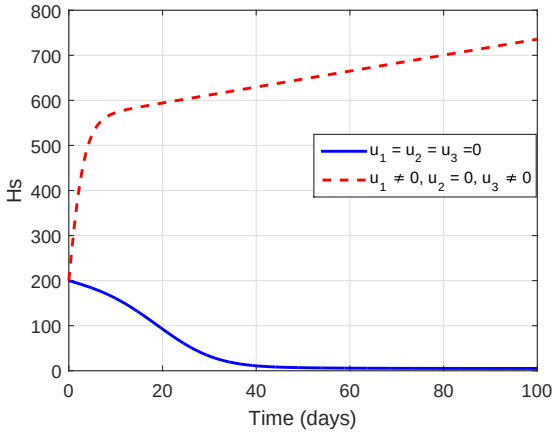


(e)

Figure 1: Set of control combination 1.



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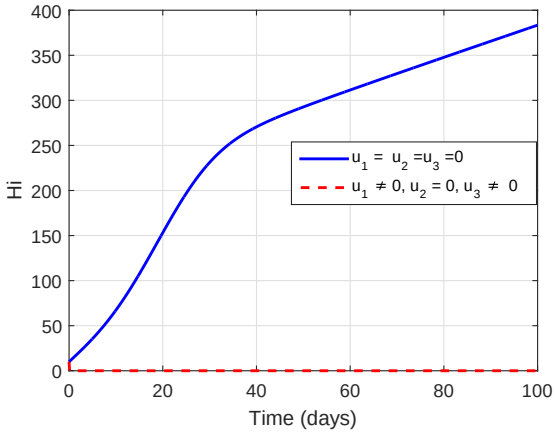

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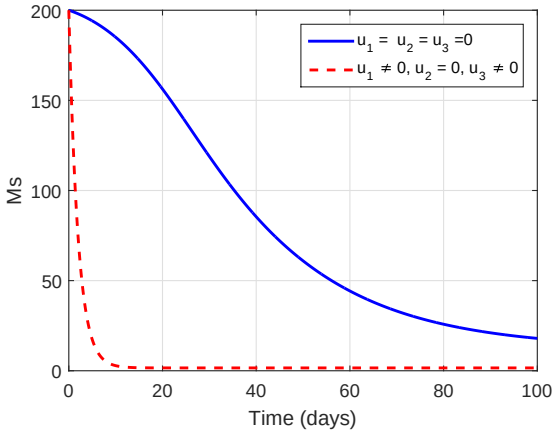

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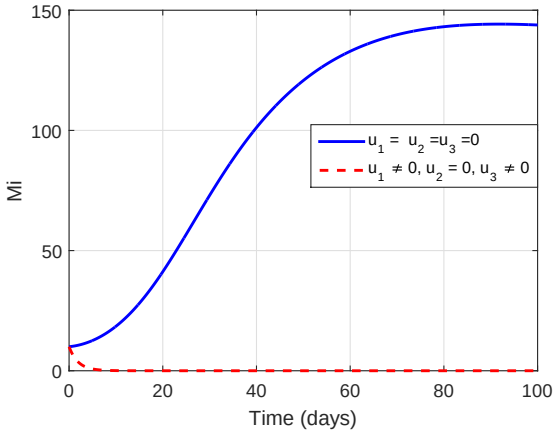

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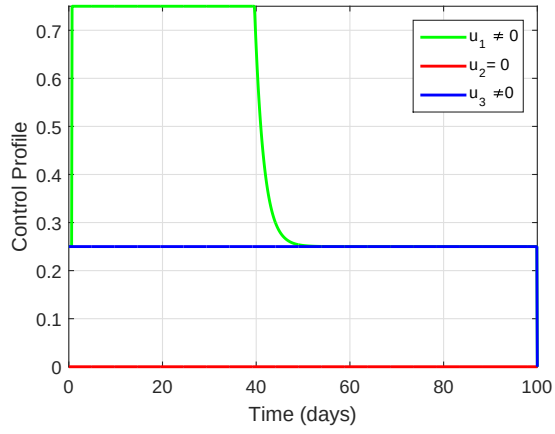

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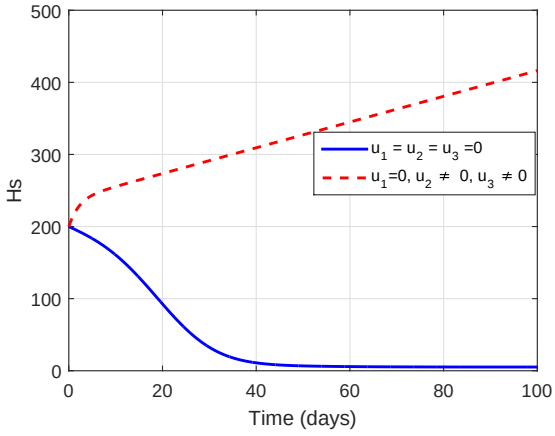
Figure 2: Set of control combination 2.



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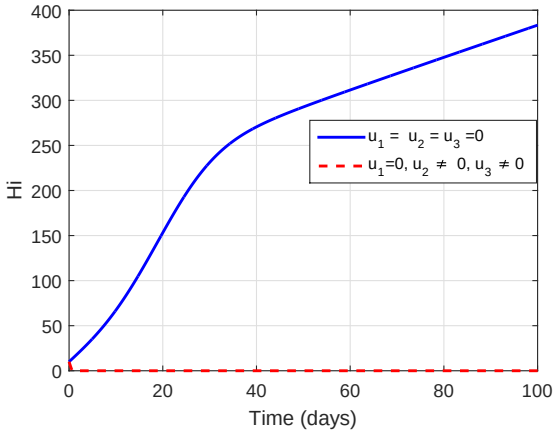

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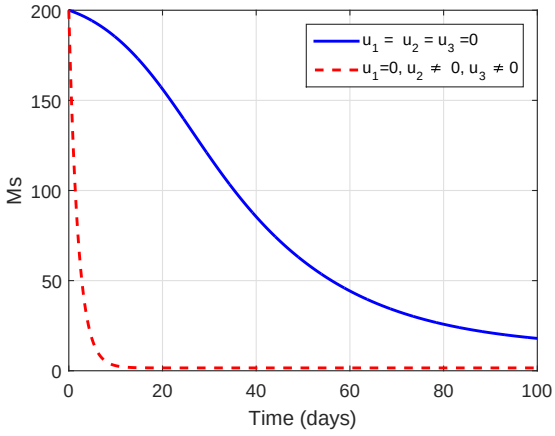

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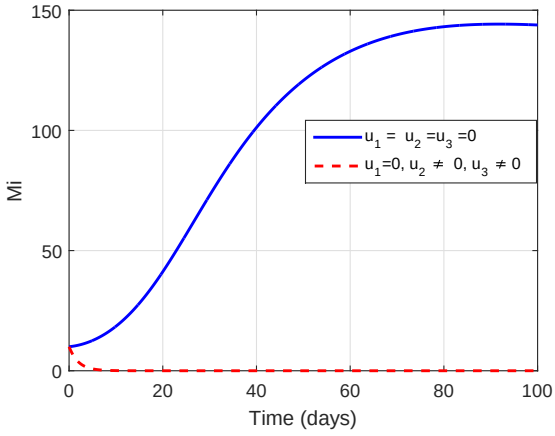

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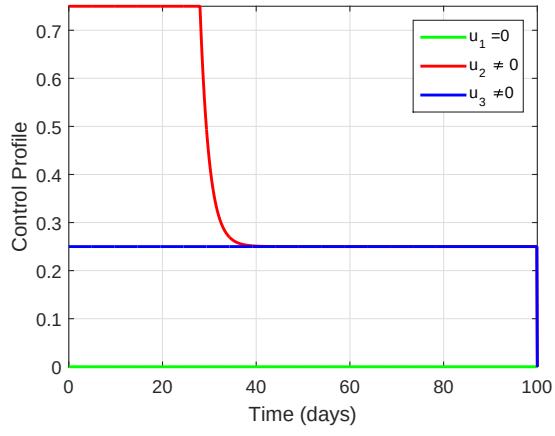

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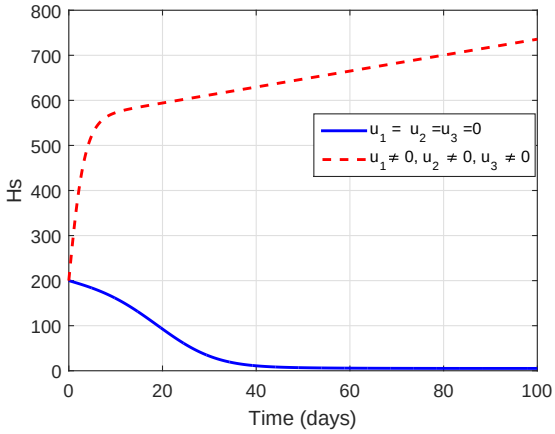
Figure 3: Set of control combination 3.



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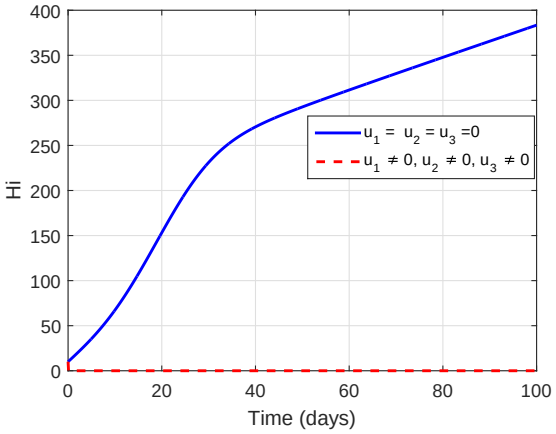

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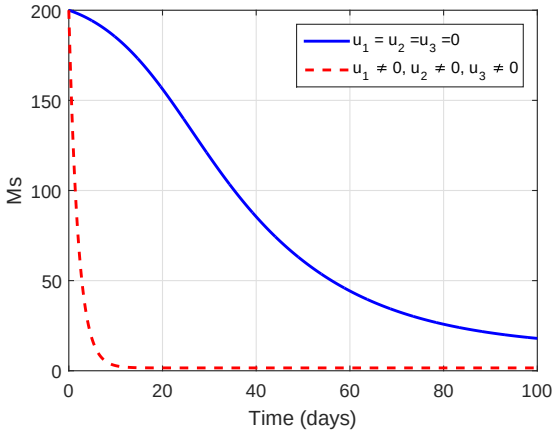

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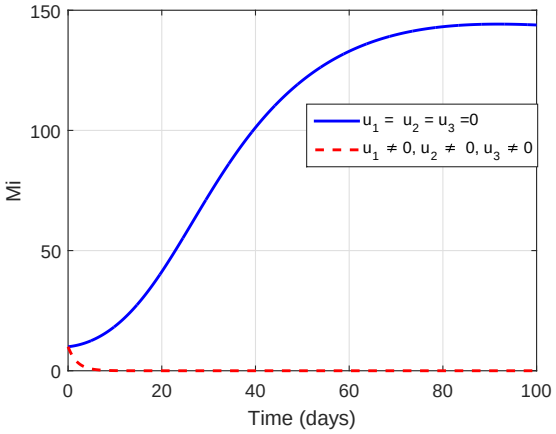

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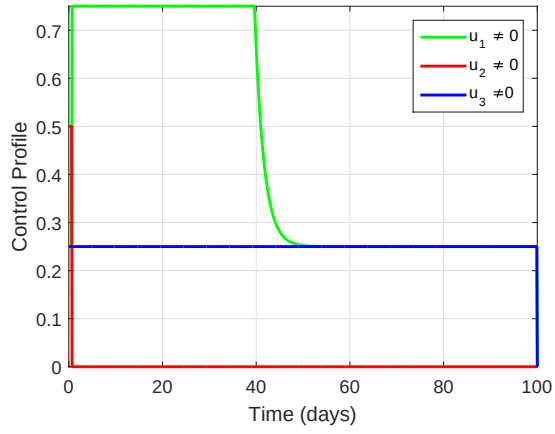

(e)

Figure 4: Set of control combination 4.

7 Conclusion

The aims of this work to formulate and analyzed the transmission dynamics of Zika virus model. Initially, the model basic results are obtained and discussed. Stability analysis of disease free endemic cases are presented and discussed. We found that, the Zika virus can be eliminated from the population if we brought the basic reproduction number less than unity. If the basic reproduction number exceeds than unity, then, the disease in the population will remain and may spread. Further, we apply three controls, we use three control variables, which are $u_1(t)$ reducing contact rate between mosquitoes and humans by wearing long sleeve shirts, big trowsers,stay in places with screen window to keep the mosquito outside, sleep under Mosquito bed net. $u_2(t)$ the transmission rate from mosquitoes to humans is reduced by increasing the auto immunity. $u_3(t)$ insecticides should be use to eradicate the mosquito, the death rate of mosquitoes should be increased, and formulate the optimal control system. The necessary results involved in the computation of optimality system are derived and discussed. The numerical solution of the optimality system in connection with system (no control) is obtained by using different set of control combinations, and we concluded that the Zika virus can be eliminated from the population if we apply the three controls at the same time.

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