Available online at www.cuijca.com

INTERNATIONAL JOURNAL OF COMPUTATIONAL ANALYSIS

Vol (5), No. (1), pp. 55-66

Study of Nonlinear Fractional Order Delay Problem Under Mittag-Leffler Power Law

*Gauhar Ali[1], Amjid Ali[2], Faiz Muhammad Khan[2] and Zia Ullah Khan[*2]*

[1]*Department of Mathematics, University of Malakand, Chakdara Dir (lower), Khyber Pakhtunkhawa, Pakistan.*

[2]*Department of Mathematics and Statistics, Univeristy of Swat, KP Pakistan, Khyber Pakhtunkhawa, Pakistan.*

**Corresponding Author: zia_khan@hotmail.com*

Keywords:

Fractional ABC derivative; Existence and stability analysis; Krasnoselskii fixed point theorem; Pantograph fractional differential equation.

Mathematics Subject classifications 2010:

26A33; 34A08; 34H15.

A B S T R A C T

In the current paper, we have studied fractional order differential equation containing Atangana-Baleanu fractional derivative in the sense of Caputo. We have studied the existence of the solution of the concerned problem with the help of using the tools of fixed point theory. Apart from this the authors have investigated the uniqueness of the solution as well. With all these we have also analysed the stability of the solution for which we have used the Ulam-Hyer's and generalized Ulam stability. We have derived all the conditions for the aforementioned work. To validate the results we derived have been illustrated through the given examples. For the existence of the solution the authors have utilized Banach and Krasnoselskii's theorems.

1 Introduction

The subject of Fractional Differential Equations (FDEs) is the main focus for researchers, due to its direct application in the diverts fields of science, such as thermodynamics, dynamics, bio-physics, memory effect, bio-medical, electrostatics, computer networking, economics, signal processing, and control theory, (see [1-5]) in the references therein. Arbitrary order derivatives is more reliable, globally in nature and high degree of freedom as compared to conventional derivatives, (see [6-11]). The researchers used various tools of fixed point theory and non-linear analysis to develop the proposed solutions for FDEs and well explored the theory up-to large extend. In this regards many articles and books have been published by researchers, for detail study we refer [12-15]. A type of FDEs in which the rate of change dependent on the previous time is known as Fractional Delay Differential Equations (FDDEs). The FDDEs provides unsurpassed techniques to modelling of natural phenomena. Proportional type delay DEs is one of the main type of DDEs, it has direct application natural problem, technological control, dynamical system and their uses [16, 17]. The DDEs have gain the attention of researchers, because of its wide range applications in dynamical systems, thermodynamics, electro-dynamics, automatic control system, hydraulic network systems, economy, transmission lines, biology and other dynamical systems (see[18]).

There are numerous types of Fractional Differential Operators (FDOs), which offer a wide selection of dishes to researchers to pick the one, which will precisely describe the circumstance. The derivative with singular kernel are immensely used and investigated by several researchers, see [19-21]. Sometime the FDOs with singular kernel determine the dynamics with nonlocal conditions. To overcome this problem, researchers introduce new FDO with singular kernel. In 2016, Caputo and Fabrizo introduced fractional differential operator with exponential functions. In the successive years, Atangana and Baleanu generalized the said derivative by replacing the exponential function by Mittage-Leffler function in the concepts of Caputo, Riemann-Liouville and was name ABC and ABR-derivatives. These are reliable fractional differential operators, to explore well the various real-world problems [22-28].

One of the salient feature of qualitative analysis of FDEs is stability analysis. Which play a foremost rule in the investigation of non-linear FDEs and is essential for the optimization and numerical point of view. It is an interesting area of research for the study of many engineering and physical problems. Although for differential, integral and functional equations, there are numerous types of stabilities discussed in the present literature, such as asymptotic stability, Exponential stability, Ullam-Hyers stability, Mittage-Leffler stability, Lyapunov stability [29-32]. Ullam-Hyers (UH) is the most reliable stability among these. Which was introduced in 1941, after the famous correspondence among the Ullam [32] and Hyers [33]. Later on, which was further generalized by researchers to generalized UH (GUH), (see [34, 35]). In 1970, Rassias modified UH stability to a more general form known as Ullam-Hyers-Rassias (UHR) stability and generalized UHR (GUHR) stability, we refer [36] to the readers. FDEs in sense of singular fractional differential operators are well studied for the stability analysis and existence of solutions [37-41]. Motivated from the existence literature of aforementioned FDEs, we studied the concerned fractional DDEs

$$\begin{cases} {}^{ABC}D_0^{\eta}\vartheta(t) = f\left(t, \vartheta(t), \vartheta(\delta t)\right),\ \vartheta(\delta t), \\ 0 < \eta \leq 1,\ 0 < \delta < 1,\ \vartheta(0) = \vartheta_0 \end{cases} \tag{1}$$

where ${}^{ABC}D_0^{\eta}$ represent ABC derivative, a continuous function $f: \mathcal{J} \times R \times R \to R$ and $\mathcal{J} = [0, T]$. Traditional class of proposed equations are well explored and furnished for existence of solutions and analysis regards to stability. In this manuscript, we studied the novel type of pantograph FDDEs in concepts of ABC arbitrary order derivative for existence theory and stability analysis. With the help of Kransnosilskii's theorem and UH type stability, we discussed the conditions for the existence of solutions and results regards to stabilities. In order to justify the desired results, the author's Stephen some illustrative examples.

2 Preliminaries

This portion is committed to known results, lemmas and definitions that are needed for further correspondence in this manuscript. Let $C[\mathcal{J}, R]$ be a Banach space with the norm define as $\| \vartheta \| = \max_{t \in \mathcal{J}} |\vartheta(t)|$. The concerned space will be used onward.

Definition 1 [22, 28]. Let $\vartheta \in H^1(a,b), a < b$ and $0, < \eta < 1$. The arbitrary order ABC operator of ϑ with order η in "Caputo" sense is given by

$$ {}^{\mathrm{ABC}}{}_{a}\mathrm{D}_t^{\eta}\vartheta(t) = \frac{\mathbb{B}(\eta)}{1-\eta}\int_a^t \vartheta'(s)\mathrm{E}_{\eta}\left(\frac{-\eta(t-s)^{\eta}}{1-\eta}\right)ds. \tag{2}$$

Similarly, "Riemann-Liouville" derivatives is define as

$$ {}^{ABR}{}_{a}\mathrm{D}_{t}^{\eta}\vartheta(t)=\frac{\mathbb{B}(\eta)}{1-\eta}\frac{d}{dt}\int_{a}^{t}\vartheta(s)\mathrm{E}_{\eta}\left(\frac{-\eta(t-s)^{\eta}}{1-\eta}\right)ds, \tag{3} $$

where $\mathbb{B}(\eta)>0$ is "normalization function with the property $\mathbb{B}(0)=\mathbb{B}(1)=1$ and E_{η} is the Mittag-Liffler function".

Definition 2 [22, 28]. The η order AB fractional integral of ϑ is defined as

$$ {}^{AB}{}_{a}\mathrm{I}_{t}^{\eta}\vartheta(t)=\frac{1-\eta}{\mathbb{B}(\eta)}\vartheta(t)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_{a}^{t}\vartheta(s)(t-s)^{\eta-1}ds. \tag{4} $$

Lemma 1 [22, 28]. "The AB fractional integral and ABC fractional derivative of order $\eta\in(0,1]$ of the function ϑ, satisfy the following

$$ {}^{AB}{}_{a}\mathrm{I}_{t}^{\eta}\,{}^{\mathrm{ABC}}{}_{a}\mathrm{D}_{t}^{\eta}\vartheta(t)=\vartheta(t)-\vartheta(a)". $$

Theorem 1 [42]. " (Kransnosilskii's fixed point theorem) If V is a non-empty closed and convex subset of X, with operators F, G such that

- $\mathrm{F}\vartheta_1+\mathrm{G}\vartheta_2\in V,\forall\,\vartheta_1,\vartheta_2\in V$;
- F is condensing operator;
- G is compact and continues;

then there exist at least one solution $\vartheta\in V$ such that

$$ \mathrm{F}(\vartheta)+\mathrm{G}(\vartheta)=\vartheta. $$

3 Existence of Theory

In this section we analyze the existence of the problem (1).

Lemma 2 If $\mathrm{y}\in\mathcal{C}[\mathcal{J},\mathrm{R}]$, then integral representation for the problem

$$ {}^{ABC}D_{0}^{\eta}\vartheta(t)=y(t),0<\eta\le 1,t\in \mathrm{I},\vartheta(0)=\vartheta_{0} \tag{5} $$

is obtain in form of

$$ \vartheta(t)=w_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}[y(t)-y_0]+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t y(s)(t-s)^{\eta-1}ds. $$

Proof. By applying the ${}^{\mathrm{AB}}\mathrm{I}_{0}^{\eta}$ to the considered problem (5), we have

$$ \vartheta(t)=b_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}y(t)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t y(s)(t-s)^{\eta-1}ds[rgb]0.00,0.00,1.00, \tag{6} $$

using $w(0)=w_0$ and $y(0)=y_0$ in (6), we have

$$ b_0=w_0-\frac{(1-\eta)}{\mathbb{B}(\eta)}y_0[rgb]0.00,0.00,1.00, $$

by putting the value of b_0 in (6), we get

$$\vartheta(t)=\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}[y(t)-y_0]+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t y(s)(t-s)^{\eta-1}ds.$$

Corollary 1. In the light of Lemma 2, the solution of the proposed problem (1) can be expressed as

$$\vartheta(t)=\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}[f(t,\vartheta(t),\vartheta(\delta t)-f_0]+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1}ds. \quad (7)$$

Next, we are interesting in existence analysis of our proposed problem, consider the following operators

$$Fw(t)=\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\vartheta(t),\vartheta(\delta t)-f_0),$$
$$Gw(t)=\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1}ds,$$
$$Kw(t)=Fw(t)+G(t).$$

Before the proof, some assumptions are needed to be hold.

- There exist a positive constant L_f such that for any $u,\vartheta,u,\bar{\vartheta}\in\mathcal{J}$, one have $|f(t,u,\vartheta)-f(t,u,\bar{\vartheta})|\leq L_f\{|u-u|+|\vartheta-\bar{\vartheta}|\}$.
- If $\exists$ positive constants l,m and n, such that $|f(t,\vartheta(t),\vartheta(\delta t))|\leq l+m|\vartheta(t)|+n|\vartheta(\delta t)|$.

Theorem 2. If the assumption (A_1) hold and $\frac{2L_f(\Gamma(\eta)+T^{\eta})}{\mathbb{B}(\eta)\Gamma(\eta)}<1$, then the propose problem (1) has unique solution.

Proof. Consider $\vartheta,\bar{\vartheta}\in C[\mathcal{J},R]$, we have

$$\|K\vartheta-K\bar{\vartheta}\|=\max_{t\in\mathcal{J}}|K\vartheta(t)-K\bar{\vartheta}(t)|$$

$$=\max_{t\in\mathcal{J}}|[\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\vartheta(t),\vartheta(\delta t)-f_0)]$$

$$+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1}ds]$$

$$-[\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\bar{\vartheta}(t),\bar{\vartheta}(\delta t))-f_0)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t f(s,\bar{\vartheta}(s),\bar{\vartheta}(\delta s))(t-s)^{\eta-1}ds]|$$

$$\leq\max_{t\in\mathcal{J}}[\frac{(1-\eta)}{\mathbb{B}(\eta)}|f(t,\vartheta(t),\vartheta(\delta t))-f(t,\bar{\vartheta}(t),\bar{\vartheta}(\delta t))|$$

$$+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t|f(s,\vartheta(s),\vartheta(\delta s))-f(s,\bar{\vartheta}(s),\bar{\vartheta}(\delta s))|(t-s)^{\eta-1}ds]$$

$$\leq\frac{2L_f((1-\eta))}{\mathbb{B}(\eta)}\|\vartheta-\bar{\vartheta}\|+\frac{2L_fT^{\eta}}{\mathbb{B}(\eta)\Gamma(\eta)}\|\vartheta-\bar{\vartheta}\|$$

$$\leq \frac{2L_f(\Gamma(\eta)+T^{\eta})}{\mathbb{B}(\eta)\Gamma(\eta)} \parallel \vartheta-\bar{\vartheta} \parallel.$$

This shows that K is a contraction. Therefore K has a unique fixed point, which is the solution to the proposed problem(1).

Theorem 3. If the assumptions (A1), (A2) holds and $0<\frac{2L_f}{\mathbb{B}(\eta)}<1$, then the mentioned problem (1) has at least one solution.

Proof. Let $V=\{\vartheta \in X: \parallel \vartheta \parallel \leq b\}$. Since f is continuous, so F is continuous. Let $\vartheta, \bar{\vartheta}$ be arbitrary elements of V, now we have

$$\parallel F\vartheta - F\bar{\vartheta} \parallel = \max_{t\in\mathcal{J}} |F\vartheta(t) - F\bar{\vartheta}(t)|$$

$$= |[\vartheta_0 + \frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\vartheta(t),\vartheta(\delta t) - f_0)] - [\vartheta_0 + \frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\bar{\vartheta}(t),\bar{\vartheta}(\delta t) - f_0)]|$$

$$\leq \frac{2L_f}{\mathbb{B}(\eta)} \parallel \vartheta - \bar{\vartheta} \parallel.$$

This implies that F is a condensing operator. Now for the compactness and continuity of G, consider for any $\vartheta \in V$, we may get

$$\parallel G\vartheta \parallel = \max_{t\in\mathcal{J}} |G\vartheta(t)|$$

$$= \max_{t\in\mathcal{J}} |\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)} \int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1} ds|$$

$$\leq \frac{m+(l+n)b}{\mathbb{B}(\eta)\Gamma(\eta)} T^{\eta}.$$

Thus G is bounded. For the propose of continuity, assume that $t_1, t_2 \in \mathcal{J}$ with a $t_1 < t_2$, we get

$$|G\vartheta(t_2) - G\vartheta(t_1)| = |\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)} \int_0^{t_2} f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1} ds$$

$$-\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)} \int_0^{t_1} f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1} ds|$$

$$\leq \frac{m+(l+n)b}{\mathbb{B}(\eta)\Gamma(\eta)} (t_2^{\eta} - t_1^{\eta}).$$

Implies that $|G\vartheta(t_2) - G\vartheta(t_1)| \to 0$ as $t_2 \to t_1$, so G is continuous. Hence by Theorem (1) our propose problem(1) has not less then one solution.

4 Stability Analysis

This segment, of the work is concern to Ulam type stability analysis. We provide some definitions and notions, which are helpful for the stability analysis of the proposed problem.

Definition 3. The solution ϑ of the mentioned problem (1) is UH stable. If we can choose a positive constant K_f such that for each $\varepsilon > 0$ and for each solution $\vartheta \in X$ of the inequality

$$|{}^{ABC}D_0^{\eta}\vartheta(t) - f(t,\vartheta(t),\vartheta(\delta t))| \leq \varepsilon,\ t \in \mathcal{J}, \tag{8}$$

we have a unique solution $\vartheta^* \in X$ of the consider problem (1), such that

$$\| \vartheta - \vartheta^* \| \leq K_f \varepsilon.$$

And is GUH stable, if one can find

$$\phi: (0,\infty) \to (0,\infty), \qquad \phi(0) = 0,$$

and for each solution ϑ of the inequality (8), we can find a unique solution ϑ^* such that

$$\| \vartheta - \vartheta^* \| \leq K_f \phi(\varepsilon).$$

Definition 4. The obtained solution of consider problem (1), i.e $\vartheta \in X$ is UHR stable under $\psi \in X$, for a positive constant K_f, such that the following result holds

$$|{}^{ABC}D_0^{\eta}\vartheta(t) - f(t,\vartheta(t),\vartheta(\delta t))| \leq \psi(t)\varepsilon, \quad \forall\ t \in [0,T], \tag{9}$$

for $\vartheta^* \in X$ be the unique solutio of the consider (1), such that

$$\| \vartheta - \vartheta^* \| \leq K_f \psi(t)\varepsilon.$$

And will be GUHR stable, if

$$\| \vartheta - \vartheta^* \| \leq K_f \psi(t).$$

Remark 1. $\vartheta \in X$ be the solution for the inequality (8), iff if there exist a function depending on ϑ is $\beta \in C[0,T]$ and for each $t \in \mathcal{J}$

- $|\beta(t)| \leq \varepsilon$;
- ${}^{ABC}D_0^{\eta}\vartheta(t) = f(t,\vartheta(t),\vartheta(\delta t)) + \beta(t)$.

Remark 2. Let $\vartheta \in X$ will be the solution of the inequality (9), iff if we have a function depending on ϑ is $\beta \in C[0,T]$ and $\forall t \in \mathcal{J}$

- $|\beta(t)| \leq \varepsilon\psi(t)$;
- ${}^{ABC}D_0^{\eta}\vartheta(t) = f(t,\vartheta(t),\vartheta(\delta t)) + \beta(t)$.

Lemma 3. In-view of above Remarks, the solution of corresponding problem

$$\begin{cases} {}^{ABC}D_0^{\eta}\vartheta(t) = f\left(t,\vartheta(t),\vartheta(\delta t)+\beta(t)\right),\ \vartheta(\delta t), \\ 0<\eta\leq 1,\ 0<\delta<1,\ \vartheta(0)=\vartheta_0 \end{cases} \tag{10}$$

satisfies the following

$$|\vartheta(t)-\mathbf{H}(t,\vartheta(t),\vartheta(\delta t))|\leq K_{\eta,T}\varepsilon,\ \forall t\in[0,T], \tag{11}$$

with

$$\mathbf{H}(t,\vartheta(t),\vartheta(\delta t) = \vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\vartheta(t),\vartheta(\delta t)-f_0)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}$$
$$\int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1}ds$$

and

$$K_{\eta,T}=\frac{\Gamma(\eta)+T^{\eta}}{M(\eta)\Gamma(\eta)}.$$

Proof. With the help of Lemma 2, (10) becomes

$$\vartheta(t)=\vartheta_0+\frac{(1-\eta)}{\mathbb{B}(\eta)}(f(t,\vartheta(t),\vartheta(\delta t)-f_0)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t f(s,\vartheta(s),\vartheta(\delta s))(t-s)^{\eta-1}ds$$
$$+\frac{(1-\eta)}{\mathbb{B}(\eta)}(\beta(t)-\beta_0)+\frac{\eta}{\mathbb{B}(\eta)\Gamma(\eta)}\int_0^t \beta(s)(t-s)^{\eta-1}ds$$

which implies that

$$|\vartheta(t)-\boldsymbol{H}(t,\vartheta(t),\vartheta(\delta t))|\leq K_{\eta,T}\varepsilon.$$

Theorem 4. Under the assumption (A_1), the desired solution of concerned problem (1) is UH and GUH stable, if $1\neq K_{\eta,T}$.

Proof. In-view of Lemma 3, if ϑ and ϑ^* are any solution and unique solution respectively for consider problem (1), such that

$$|\vartheta(t)-\vartheta^*(t)|=|\vartheta(t)-\mathbf{H}(t,\vartheta^*(t),\vartheta^*(\delta t))|$$
$$=|\vartheta(t)-\mathbf{H}(t,\vartheta(t),\vartheta(\delta t))+\mathbf{H}(t,\vartheta(t),\vartheta(\delta t))+\mathbf{H}(t,\vartheta^*(t),\vartheta^*(\delta t))|$$
$$\leq|\vartheta(t)-\mathbf{H}(t,\vartheta(t),\vartheta(\delta t))|+|\mathbf{H}(t,\vartheta(t),\vartheta(\delta t))+\mathbf{H}(t,\vartheta^*(t),\vartheta^*(\delta t))|$$
$$\leq K_{\eta,T}\varepsilon+2L_fK_{\eta,T}\parallel\vartheta-\vartheta^*\parallel$$

which further yields that

$$\parallel\vartheta-\vartheta^*\parallel\leq\frac{K_{\eta,T}}{1-2L_fK_{\eta,T}}\varepsilon.$$

Expressing by $K_f=\frac{K_{\eta,T}}{1-2L_fK_{\eta,T}}$, then the propose problem(1) is UH stable. Also, if $\phi(\varepsilon)=\varepsilon$, then the concerned solution is GUH stable.

Lemma 4. The following inequality holds for consider problem (10):

$$|\vartheta(t) - \mathbf{H}(t, \vartheta(t), \vartheta(\delta t))| \leq K_{\eta,T}\varepsilon\Psi(t), \qquad \text{forall } t \in \mathcal{J},$$

Proof. we omit the proof, just similar to that of Lemma 3 by using Remark 2.

Theorem 5. The solution of our propose model (1) is UHR and GUHR stable, if $K_{\eta,T} \neq 1$, assumption (A_1) hold along with Lemma 4.

Proof. Just like Theorem 4, we can derive the required results.

5 Stability Analysis

In this section, we steepen some examples for the illustration of our main work.

Example 1 Consider the following initial value problem of "pantograph" type FODEs under ABC derivative

$$\begin{cases} {}^{ABC}\mathbf{D}_t^{\frac{1}{2}}\vartheta(t) = \dfrac{t^2}{25} + \dfrac{\sin|\vartheta(t)| + \sin\left|\vartheta\dfrac{t}{2}\right|}{50 + t^3}, \ t \in [0,1], \ \vartheta(0) = 0, \end{cases} \tag{12}$$

Clearly $T = 1$ and $f(t, \vartheta(t), \vartheta(\frac{1}{2}t)) = \frac{t^2}{25} + \frac{sin|\vartheta(t)|+sin|\vartheta(\frac{t}{2})|}{50+t^3}$ is continuous function $\forall\ t \in [0,\ 1]$. Further let $\vartheta, \bar{\vartheta} \in \mathcal{C}[\mathcal{J}, R]$, then we gets

$$|f(t, \vartheta(t), \vartheta(\frac{1}{2}t)) - f(t, \bar{\vartheta}(t), \bar{\vartheta}(\frac{1}{2}t))| = |[\frac{t^2}{25} + \frac{\sin|\vartheta(t)| + \sin|\vartheta(\frac{t}{2})|}{50 + t^3}]$$

$$-[\frac{t^2}{25} + \frac{\sin|\bar{\vartheta}(t)| + \sin|\bar{\vartheta}(\frac{t}{2})|}{50 + t^3}]| \leq \frac{1}{50}[|\vartheta(t) - \bar{\vartheta}(t)| + |\vartheta(\frac{1}{2}t) - \bar{\vartheta}(\frac{1}{2}t)|].$$

Hence from above, one has $L_f = \frac{1}{50}$, and $\eta = \frac{1}{2}$. Further,

$$|f(t, \vartheta(t), \vartheta(\frac{1}{2}t))| = |\frac{t^2}{25} + \frac{\sin|\vartheta(t)| + \sin|\vartheta(\frac{t}{2})|}{50 + t^3}| \leq \frac{1}{25} + \frac{1}{50}|\vartheta(t)| + \frac{1}{50}|\vartheta(\frac{1}{2}t)|.$$

Here, $l = \frac{1}{25}$, $m = \frac{1}{50}$, $n = \frac{1}{50}$ and $T = 1$.

Now

$$\frac{T^\eta + 2L_f(\Gamma(\eta))}{\Gamma(\eta)\mathbb{B}(\eta)} = 0.0469.$$

Thus, Theorem (2) is satisfied. Hence, the consider example (12) has a unique solution. Further, $\frac{2L_f}{\mathbb{B}(\eta)} = 0.03$. Therefore, Theorem (3) holds. Thus, (12) has at least one solution. Furthermore, proceed to verify stability results, we see that $K_{\eta,T} = 1.1731 \neq 1$. Hence the solution of mentioned problem (12) is UH stable and consequently GUH stable. Also in the same way one can analyzed the UHR and GUHR stabilities analysis for (12).

Example 2 Let us consider the Pantograph type problem of ABC FODEs as

$$\begin{cases} {}^{ABC}D_t^{\frac{3}{7}}\vartheta(t)=\dfrac{t+e^{2t}}{10}+\dfrac{e^{3t}\sin\left|\vartheta(t)\right|}{35+t^2}+\dfrac{3t^2\sin\left|\vartheta\left(\frac{t}{3}\right)\right|}{35},\ t\in[0,1],\ \omega(0)=0, \end{cases} \tag{13}$$

$f(t,\vartheta(t),\vartheta(\frac{1}{3}t))=\frac{t+e^{2t}}{10}+\frac{e^{3t}\sin|\vartheta(t)|}{35+t^2}+\frac{3t^2\sin|\vartheta(\frac{t}{3})|}{35}$ is continuous function for all $t\in[0,\ 1]$. Further let $\vartheta,\bar{\vartheta}\in C[\mathcal{J},R]$, then consider that

$$|f(t,\vartheta(t),\vartheta(\frac{1}{3}t))-f(t,\bar{\vartheta}(t),\bar{\vartheta}(\frac{1}{3}t))|=|\{\frac{t+e^{2t}}{10}+\frac{e^{3t}\sin|\vartheta(t)|}{35+t^2}+\frac{3t^2\sin|\vartheta(\frac{t}{3})|}{35}\}$$
$$-\{\frac{t+e^{2t}}{10}+\frac{e^{3t}\sin|\vartheta(t)|}{35+t^2}+\frac{3t^2\sin|\vartheta(\frac{t}{3})|}{35}\}|\le\frac{1}{35}\{|\vartheta(t)-\bar{\vartheta}(t)|+|\vartheta(\frac{1}{3}t)-\bar{\vartheta}(\frac{1}{3}t)|\}.$$

Thus from above, one has $L_f=\frac{1}{35}$ and $\eta=\frac{3}{7}$. And also, we have

$$|f(t,\vartheta(t),\vartheta(\frac{1}{3}t))|=|\frac{t+e^{2t}}{10}+\frac{e^{3t}\sin|\vartheta(t)|}{35+t^2}+\frac{3t^2\sin|\vartheta(\frac{t}{3})|}{35}|\le\frac{1}{10}+\frac{1}{35}|\vartheta(t)|+\frac{1}{35}|\vartheta(\frac{1}{3}t|.$$

Where $l=\frac{1}{10}, m=\frac{1}{35}, n=\frac{1}{35}$ and $T=1$. Further,

$$\frac{2L_f(\Gamma(\eta)+T^{\eta}}{\mathbb{B}(\eta)\Gamma(\eta)}=\frac{22(\Gamma(\frac{3}{7})+1)}{490\Gamma(\frac{3}{7})}<1.$$

Hence, the deserted conditions of Theorem 2 are satisfied. Which granted the uniqueness of solution for concerned problem (13). Further

$$\frac{2L_f}{\mathbb{B}(\eta)}=\frac{11}{245}<1.$$

Thus, also the condition mandatory for Theorem 3 holds. Therefore, (13) has at least one solution. Furthermore, we computed that $K_{\eta,T}\neq 1$, which elaborates that concerned problem (13) is both generalized and Hyers-Ulam stable. In same way it is easy to prove the conditions for UHR and GUHR stability.

6. Concluding Remarks

In this paper we have developed the constraints for the existence and stability of the solution of the problems containing the Atangana-baleanu fractional differential operator in the sense of Caputo. Apart from this, we have used the tools of fixed point theory such as Krasnoselskii and Banach theorems to derive the conditions for the problem studied in this paper. Furthermore, to analyze the stability of the problems we have provided the Ulam and generalized Ulam stability. For the verification of the results we also provided examples and calculated the conditions under which the problems the unique and stable solution of the problems exists.

Competing interest: The authors declare that they have no competing interest regarding this work.

References

[1] M.K. Ishteva, properties and application of the Caputo Fractional operator, Deptt. Math. Univ. Karlsruhe, 2005.

[2] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific Publishing. USA, 1964.

[3] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley New York, 1993.

[4] A. Ali, K. Shah, R.A. Khan, Existence of positive solution to a class of boundary value problems of fractional differential equations, Compu. Methods Diff. Equ. 4 (2016) 19-29.

[5] A. Ali, et al. Existence and stability of solution to a toppled systems of differential equations of non -integer order, Bound. Value. Probl. 2017 (2017) 11.

[6] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, Academic Press, New York, 1999.

[7] M. Caputo, Linear Models of dissipation whose Q is almost frequency independent, Int. Jou. Geo. Sci. 13 (1967) 529-539.

[8] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[9] A.A. Kilbas, O.I. Marichev, S.G. Samko, Fractional integrals and derivatives, Gordon and Breach, Switzerland, 1993.

[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland mathematics Studies, Elsevier, Amsterdam, 204 2006.

[11] V. Lakshmikantham, S. Leela, J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, UK, 2009.

[12] L. Cai, J. Wu, Analysis of an HIV/AIDS treatment model with a nonlinear incidence rate, Chaos. Soliton. Frat. 41 (2009) 175-182.

[13] R.C. Wu, X.D. Hei, L.P. Chen, Finite-time stability of fractional-order neural networks with delay, Commun. Theor. Phys. 60 (2013) 189-193.

[14] A. Nanware, D.B. Dhaigude, Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions, J. Nonlinear Sci. Appl. 7 (2014) 246-254.

[15] R.P. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, Adv. Differ. Equ. 2009 (2009) 1-47.

[16] M. Buhmann, A. Iserles, Stability of the discretized pantograph differential equation, Mathematics of Computation, 60(202) (1993) 575-589.

[17] J. Cao, J. Lu, Adaptive synchronization of neural networks with or without time-varying delay, Chaos: An Interdisciplinary Journal of Nonlinear Science, 16(1) (2006) 013133.

[18] R.L. Magin, Fractional calculus models of complex dynamics in biological tissues, Comput. Math. Appl. 59 (2010) 1586-1593.

[19] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, J. Nonl. Anal. 71 (2009) 2391–2396.

[20] K. Shah, R.A. Khan, Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions, Differ. Equ. Appl. 7 (2015) 245–262.

[21] R.A. Khan, K. Shah, Existence and uniqueness of solutions to fractional order multi-point boundary value problems, Commun. Appl. Anal. 19 (2015) 515–526.

[22] A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel, Theor, Appl. Heat Transfer Model, Thermal Science. 20 (2016) 763-769.

[23] J.D. Djida, A. Atangana, I. Area, Numerical computation of a fractional derivative with non-local and non-singular kernel, Math. Model. Nat. Phenomena. 12 (2017) 4-13.

[24] O. Algahtani, Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order Allen Cahn model, Chaos Solitons Fractals, 89 (2016) 552–559.

[25] J.C. Trigeassou et al. A Lyapunov approach to the stability of fractional differential equations, Signal Process, 91(3) (2011) 437-445.

[26] G. Lijun, D. Wang, G. Wang, Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects, Appl. Math. Comput. 268 (2015) 186-200.

[27] I. Stamova, Mittag-Leffler stability of impulsive differential equations of fractional order, Q. Appl. Math. 73(3) (2015) 525-535.

[28] S.M. Ullam, Problems in Modern Mathematics, Science Editors, Wiley, New York 1940.

[29] D.H. Hyers, On the stability of the linear functional equation Proc. Natl. Acad. Sci. USA, 27(4) (1941) 222-224.

[30] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1940.

[31] S.M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1960.

[32] T.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. 72 (1978) 297-300.

[33] A. Ali, K. Shah, T. Abdeljawad, Study of implicit delay fractional differential equations under anti-periodic boundary conditions, Advances in Difference Equations, 2020(1) (2020) 1-16.

[34] A. Coronel-Escamilla, J.F. Gómez-Aguilar, A novel predictor-corrector scheme for solving variable-order fractional delay differential equations involving operators with Mittag-Leffler kernel, Discrete Contin. Dyn. Syst., Ser. S, 13(3) (2020) 561-574.

[35] K. Balachandran, S. Kiruthika, J.J. Trujillo, Existence of solution of nonlinear fractional pantograph equations, Acta Math. Sci. 33(3) (2013) 712-720.

[36] E. Tohidi, A.H. Bhrawy, K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, Appl. Math. Model. 37(6) (2013) 4283-4294.

[37] M. Iqbal, K. Shah, R.A. Khan, On using coupled fixed point theorems for mild solutions to coupled system of multi-point boundary value problems of nonlinear fractional hybrid pantograph differential equations. Math. Methods Appl. Sci. 44 (2019) 1-14.

[38] T.A. Burton, A fixed point theorem of Krasnoselskii, Appl. Math. Lett. 11 (1998) 85-88.