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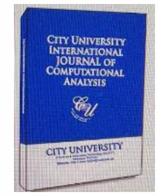

Caputo-Fabrizio Fractional Model of Electro-Osmotic Flow of Walters'-B Fluid in the Presence of Diffusion-Thermo: Exact Solution via Integral Transform

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A B S T R A C T

Electrically conducted viscoelastic fluids have recently caught the attention of scientists and engineers due to their extensive applications in various sectors of research and engineering. It has great importance in cancer therapy (hyperthermia), magnetic resonance imaging (MRI), medication administration, and magnetic refrigeration (MR). The core objective of the present manuscript is to find the exact solution of the fractional convective flow of Walters'-B fluid. The effects of thermal radiations, magnetic field, electro-osmosis, and diffusion thermo have been considered in the present phenomenon. With the help of relative constitutive equations, the governing equations of the present phenomenon have been modeled in terms of second-order partial differential equations. To obtain the closed-form solution for velocity, temperature, and concentration equation, the Laplace transform technique has been implemented. To check the influences of various inserted parameters on fluid, graphs have been plotted. It is very important to mention that electro-osmotic and Walters'-B fluid parameters decline the profile of velocity.

1 Introduction

Right now, electro-assimilation accomplished the focal consideration of numerous analysts and researchers in view regarding its importance in microscale and nano implements having some routine and day-to-day applications in systematic science, the machine-driven procedure also in medication. The phenomena of electro-osmosis allude to the motion of charged ions found in fluid in a permeable material because of a pragmatic electric field. This idea of creating motion by taking the help of the outer electric field starts two or three several years prior at the point when Reuss [1], inspected this marvel cautiously, utilizing clay. Recently, a lot of scientists and analysts contributing tentatively, mathematically, and hypothetically for a better understanding of the concept of electro-osmosis. The principal component of electro-osmosis comprises the exodus of the ions from one place to another, in simple words, one can say that movement of anion towards the anode and also the movement of cation towards cathode is the main concept behind the process of electro-osmosis [2]. The hypothesis of hydration and model of Spiegler frictional [3,4] are the speculations castoff to depict the phenomena of electro-osmosis. Employments on large scale such as microfluidics appliances assume a significant function in numerous arenas, similar to substantial intermingling, clinical investigative, energy reaping, and synthetic investigation. The main fundamental problem of the appliances such as microfluidics is that in which way, we can move fluid which is generally increased through pressure-driven flows. The Electro-osmosis concept is comprehensively used as an option for siphoning the fluids in microchannel. The primary concern that varieties electro-osmosis an ideal technique for siphoning fluids in microchannel is that, speed of the motion of electro-osmotic velocities is liberated from the channel estimations. Electro-osmotic flows of other liquids which are other than Newtonian liquids to which we called non-Newtonian, for instance, blood, polymeric, protein plans, colloidal postponements, and the polymeric courses of action are critical. Freshly, various examinations of electro-osmotic flows of other liquids like non-Newtonian liquids have been accounted for with various kinds of native models, such as the models of power-law [5-8] and models of Maxwell [9,10]. As aware of the importance of electro-osmotic flow Ali *et al.* (11) studied the time-fractional analysis of the electro-osmotic flow of Walters'-B fluid and find the exact solutions for velocity, temperature, and concentration profile. The results obtained from this research work are of great importance. The obtained results of the current paper were justified by plotting the graphs. Knowing about the importance of the electro-osmotic flow of non-Newtonian fluids Saqib *et al.* [12] find the exact analysis of the non-linear electro-osmotic flow fractional Maxwell model using nanoparticles. They also find Nusselt number and skin friction in their research work. The authors also verified their mathematical work by plotting the graphs for different physical parameters. The results of the current work are very meaningful and fruitful.

The history of fractional calculus is very rich, which started hundreds of years ago. But from the last thirty years, fractional calculus shows marvellous development and succeeded to achieve the worthy attention of numbers of researchers, scientists, and engineers. Many mathematicians of their time like Leibniz, Laplace, Holmgren, Hardy, Fourier, Riesz, and Caputo contributed much to this field [13]. One can say that nowadays fractional is the need of the modern world. It shows its remarkable role in every field of life for example the physics related to bio, mechanics, physics, engineering, electrochemistry, bioengineering, mechatronics, and viscoelasticity [14-17]. Furthermore, from the study of fractional calculus, it is noted that it involves different operators which are developed by different mathematicians in which some of the popular and the most used fractional operators are listed which include Caputo-Kober, Rieman-Liouville, Caputo-Hadamard, Caputo, and Caputo-Fabrizio [18-22]. At first, Riemann-Liouville was the most utilized definition yet there are a few issues in the uses of Riemann-Liouville, for example, the main issue was that the derivative of a constant isn't zero. To get around this deficiency of the Riemann-Liouville derivative, Caputo inaugurate a revamp rendition of fractional derivatives, but the kernel was found singular. Mustering out the remonstrance of Caputo derivative (CD), Caputo and Fabrizio innovate a newfangled delineation by the name of, Caputo-Fabrizio derivative (CFD) having non-singular kernel, which was omphalos on exponential function [23]. Recently Ali *et al.* [24] examined modelling of transient MHD Brinkman nanofluid exploit the concept of CF fractional derivative. In the above-stated paper, the authors scrutinize the impact of nanoparticles of Cu on the free convection MHD transient movement of C6H9NaO7 Brinkman nanofluid lying on a vertical plate along with time-pendent concentration, temperature, and velocity. The exact solutions of the under-discussed paper obtained were of great importance and value. In another paper by Ali *et al.* [25], the authors work on the impacts of various-shaped nanoparticles on the accomplishment of Engine-Oil and Kerosene-Oil along with Brinkman as a base fluid. In the above-stated paper, the authors utilized the aspiration of Caputo-Fabrizio time derivative to transform the classical model into a generalized model. In continuation and knowing the importance of Caputo-Fabrizio fractional time derivative in 2019 Ali *et al.* [26] published his paper heat transfer analysis of generalized Jeffery nanofluid in a rotating frame. A mathematical tool called the Laplace transform technique is utilized for obtaining the closed-form solutions for velocity and temperature field. The aspiration of Caputo-Fabrizio time-fractional derivative is used by Saqib *et al.* [27] to find closed-form results for generalized Jeffrey fluid having free convection flow. Some other important and meaningful research work regarding Caputo-Fabrizio fractional derivative can be seen in [28, 29].

Bearing in the mind of the above-addressed literature, the authors of the manuscript did not find any study related to the fractional electro-osmotic flow of Walter's-B fluid with the effects of diffusion thermos. Therefore, to fill this gap in the literature the authors assumed time-dependent, the incompressible flow of viscoelastic Walter's-B fluid together with the influence of diffusion-thermos and thermal radiation. By making use of relative constitutive equations, the non-local mathematical model has been developed. The non-local mathematical model has been transformed to a fractional model by incorporating the time-fractional derivative of Caputo-Fabrizio with an exponential kernel. The exact solution has been found for velocity, temperature, and concentration equation by employing the integral transform technique i.e., Laplace transforms technique.

2 Mathematical Model of the Problem

Assumed an incompressible flow of viscoelastic Walters'-B fluid together with the impact of diffusion thermo and electro-osmosis. The Plate is considered vertically upward in the direction of the x-axis. The fluid occupies the y, z plane. At the start, the fluid and the plate are in a static position, but after $t > 0$ the plate has been disturbed with the sudden jerk of which transmits the motion in the fluid along the x-axis. The configuration of the flow has been presented in Fig.1.

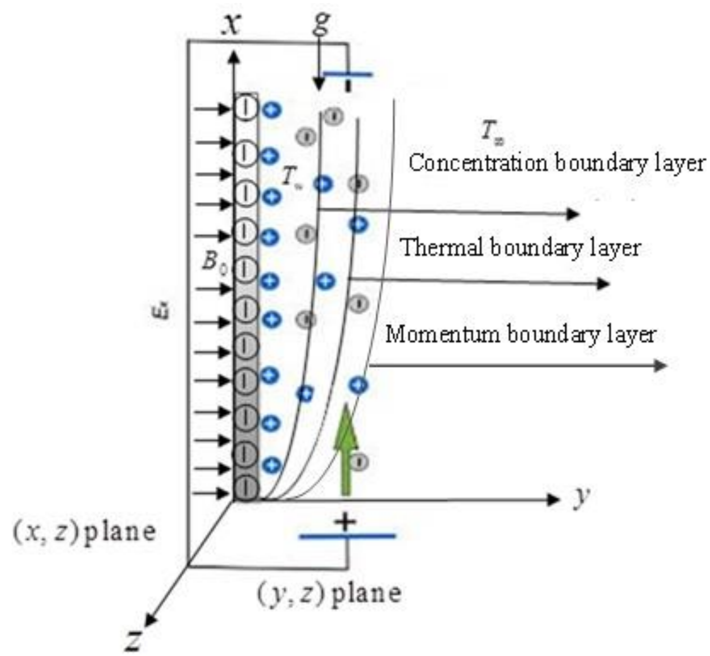

Figure.1: Configuration of the Flow

The governing equations that describes the fluid motion are presented as [11]:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{k_0}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial^2 y}\right) - \frac{\phi^{\bullet}}{k_p}\left(\nu + \frac{k_0}{\rho}\frac{\partial}{\partial t}\right)u - \frac{\sigma^{\bullet} B_0{}^2 u}{\rho} + g\beta_T\left(T - T_a\right) + g\beta_c\left(C - C_a\right) + \frac{E_x\rho_e}{\rho}. \tag{1}$$

$$\rho c_p \frac{\partial T}{\partial t} = k_1 \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_r}{\partial y}. + \frac{D_m K_T \rho}{C_s}\frac{\partial^2 C}{\partial y^2}, \tag{2}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} \tag{3}$$

and initial and boundary conditons:

$$\left.\begin{array}{lll} u(y,0)=0, & T(y,0)=T_a, & C(y,0)=C_a, \\ u(0,t)=U_0 H(t)\cos wt, & T(0,t)=T_p, & C(0,t)=C_p, \\ u(\infty,t)=0, & T(\infty,t)=T_a & C(\infty,t)=C_a. \end{array}\right\}. \tag{4}$$

In the governing equation (2), the term radiative heat flux is given as [30]:

$$q_r = -\frac{4\sigma_1^{\bullet}}{3k_2}\frac{\partial T^4}{\partial y} \ , \tag{5}$$

For expanding T^4, the Taylor series about $\left(T - T_a\right)$ has been used and ignoring the higher order terms, we get:

$$T^4 \cong 4TT_a^{\ 3} - 3T_a^{\ 4}, \tag{6}$$

Differentiating equation (6) with respect to "y" and making use of equation (5), equation (2) takes the form:

$$\rho c_p \frac{\partial T}{\partial t} = k_1\left(\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma_1^{\bullet} T_\infty^{\ 3}}{3k_1k_2}\frac{\partial^2 T}{\partial y^2}\right) + \frac{D_m k_T \rho}{C_s}\frac{\partial^2 C}{\partial y^2}, \tag{7}$$

For making the governing equations dimensionless, relative dimensionless variables are:

$$u^{\bullet} = \frac{u}{U_0}, \quad y^{\bullet} = \frac{U_0}{\nu} y, \quad t^{\bullet} = \frac{U^2{}_0}{\nu} t, \quad \theta = \frac{T - T_a}{T_p - T_a} \ , \quad \phi = \frac{C - C_a}{C_a - C_a} \ . \tag{8}$$

In the light of above stated dimensionless variables, the system of dimensionless governing equations is:

$$\Gamma_0 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - \Gamma \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial y^2}\right) - k_{eff} u + Gr\theta + Gm\phi - E_s e^{-ky}, \tag{9}$$

$$\frac{\partial \theta}{\partial t} = \Psi^* \frac{\partial^2 \theta}{\partial y^2} + Dr \frac{\partial^2 \phi}{\partial y^2}, \tag{10}$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{Sc}\frac{\partial^2 \phi}{\partial y^2}. \tag{11}$$

and intial and boundary conditions.

$$\left.\begin{array}{lll} u(y,0) = 0 & \theta(y,0) = 0 & \phi(y,0) = 0 \\ u(0,t) = H(t)\cos wt & \theta(0,t) = 1 & \phi(0,t) = 1 \\ u(\infty,t) = 0 & \theta(\infty,t) = 0 & \phi(\infty,t) = 0 \end{array}\right\} . \tag{12}$$

Here,

$\Gamma = \frac{k_0 U_0^{\ 2}}{\rho \nu^2}$ represent Walter's-B fluid parameter, $\frac{1}{k} = \frac{\nu^2 \phi}{k_p U_0^{\ 2}}$ represent permeability parameter, $Gr = \frac{\nu g \beta_T (T_p - T_a)}{U_0^{\ 3}}$ shows thermal Grashof number, $Gm = \frac{\nu g \beta_c (C_p - C_a)}{U_0^{\ 3}}$ shows mass Grashof number, $Sc = \frac{\upsilon}{D}$ shows Schmidt number, $\Pr = \frac{\mu c_p}{k_1}$ is Prandtl number, $Rd = \frac{16\sigma_1^{\bullet} T_\infty^{\ 3}}{3k_1 k_2}$ is radiation parameter,

$Df = \dfrac{D_m k_T}{C_s C_p \nu}\left(\dfrac{C_p - C_a}{T_p - T_a}\right)$ indicates Dufour number, and $Es = \dfrac{\nu E_x \varepsilon k^2 \psi_p}{U_0{}^3 \rho}$ is the electro-osmotic parameter, while $\dfrac{1}{\Psi^*} = \dfrac{\Pr}{1+Rd}$ $\Gamma_0 = 1 + \dfrac{\Gamma}{k}$, $k_{eff} = \dfrac{1}{k} + M$, $k_3 = \dfrac{k\nu}{U_0}$ are constants.

3. Caputo-Fabrizio Fractional Model

The Caputo-Fabrizio fractional operator of order is defined as [18]:

$$ {}^{CF}C_t^{\alpha} f(t) = \frac{N(\alpha)}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-t^*)}{1-\alpha}\right)\frac{\partial f(t)}{\partial t^*}dt^*, \qquad 0<\alpha<1. \tag{13}$$

Here $N(\alpha)$ is the normalization function such that $N(0)=1=N(1)$. In the light of fractional operator of Caputo-Fabrizio the classical governing equations adopt the following shape:

$$\Gamma_0\ {}^{CF}C_t^{\alpha} u(y,t) = \frac{\partial^2 u}{\partial y^2} - \Gamma\ {}^{CF}C_t^{\alpha}\left(\frac{\partial^2 u}{\partial y^2}\right) - k_{eff}u + Gr\theta + Gm\phi - E_s e^{-ky}, \tag{14}$$

$$ {}^{CF}C_t^{\alpha}\theta(y,t) = \Psi^* \frac{\partial^2 \theta}{\partial y^2} + Df\frac{\partial^2 \phi}{\partial y^2}, \tag{15}$$

$$ {}^{CF}C_t^{\alpha}\phi(y,t) = \frac{1}{Sc}\frac{\partial^2 \phi}{\partial y^2}. \tag{16}$$

4. Solution of the Fractional Model

For the exact solution of the fractional model, the Laplace transform technique has beeen used on equations (14-16), which takes the form:

$$\frac{d^2\bar{u}}{dy^2} - \Gamma_6\left(\frac{q+\Gamma_5}{q+\Gamma_7}\right) = -Gr\bar{\theta} - Gm\bar{\phi} + \frac{E_s e^{-ky}}{q}, \tag{17}$$

$$\frac{d^2\bar{\theta}}{dy^2} - \left(\frac{qa_2}{q+a_1}\right)\bar{\theta} = \frac{Df_1}{q+a_1}\exp\left(-y\sqrt{\frac{bq}{q+a_1}}\right), \tag{18}$$

$$\frac{d^2\bar{\phi}}{dy^2} - \left(\frac{qb}{q+a_1}\right)\bar{\phi} = 0, \tag{19}$$

and the Laplace transformed boundary conditions are:

$$\left.\begin{array}{lll}\bar{u}(0,q)=H(q)\dfrac{q}{q^2+w^2}, & \bar{\theta}(0,q)=\dfrac{1}{q} & \bar{\phi}(0,q)=\dfrac{1}{q} \\ \bar{u}(\infty,q)=0, & \bar{\theta}(\infty,q)=0 & \bar{\phi}(\infty,q)=0\end{array}\right\}. \tag{20}$$

Solving equations (17-19) and using transformed boundary conditions stated in equation (20), we arrived at:

$$\begin{aligned}\bar{u}(y,q)=&\frac{q}{q^2+w^2}\exp\left(-y\sqrt{\Gamma_6\left(\frac{q+\Gamma_5}{q+\Gamma_7}\right)}\right)+\left[\begin{array}{l}\dfrac{b_0(q+a_1)^2}{q\left[\delta_1q^2+\delta_2q-\delta_3\right]}\\ +\dfrac{b_1(q+a_1)^2}{q\left[\delta_6q^2+\delta_7q-\delta_8\right]}\\ -\dfrac{E_s(q+a_1)}{q\left[(q+a_7)-(q+\Gamma_5)\Gamma_8\right]}\end{array}\right]\exp\left(-y\sqrt{\Gamma_6\left(\frac{q+\Gamma_5}{q+\Gamma_7}\right)}\right)\\ &+\frac{E_s(q+a_1)e^{-ky}}{q\left[(q+a_7)-(q+\Gamma_5)\Gamma_8\right]}-\frac{\Psi_1(q+a_1)^2}{q\left[\delta_4q^2+\delta_5q-\delta_3\right]}\exp\left(-y\sqrt{\frac{bq}{q+a_1}}\right)\\ &-\frac{b_0(q+a_1)^2}{q\left[\delta_1q^2+\delta_2q-\delta_3\right]}\exp\left(\sqrt{\frac{qa_2}{q+a_1}}\right)-\frac{b_1(q+a_1)^2}{q\left[\delta_6q^2+\delta_7q-\delta_8\right]}\exp\left(-y\sqrt{\frac{bq}{q+a_1}}\right)\end{aligned} \tag{21}$$

$$\bar{\theta}(y,q)=\frac{Df_3}{q}\exp\left(-y\sqrt{\frac{qa_2}{q+a_1}}\right)+\frac{Df_2}{q}\exp\left(-y\sqrt{\frac{bq}{q+a_1}}\right), \tag{22}$$

$$\bar{\phi}(y,q)=\frac{1}{q}\exp\left(\sqrt{\frac{bq}{q+a_1}}\right). \tag{23}$$

Where

$a_0=\dfrac{1}{1-\alpha},\quad a_1=a_0\alpha,\quad a_2=\dfrac{a_0}{\Psi^*},\quad \Gamma_1=\Gamma_0a_0,\quad \Gamma_2=\Gamma a_0,\quad \Gamma_3=1-\Gamma_2,\quad \Gamma_4=k_{eff}+\Gamma_1,$

$\Gamma_5=\dfrac{a_3}{\Gamma_4},\quad \Gamma_6=\dfrac{\Gamma_4}{\Gamma_3},\quad \Gamma_5=\dfrac{a_1}{\Gamma_3},\quad \Gamma_8=\dfrac{\Gamma_6}{k^2},\quad \delta_1=a_2-\Gamma_6,\quad \delta_2=a_2\Gamma_7-a_1\Gamma_6-\Gamma_6\Gamma_5,$

$\delta_3=a_1\Gamma_5\Gamma_6,\quad \delta_4=b\Gamma_7-\Gamma_6\Gamma_5-\Gamma_6a_1,\quad \delta_5=b-\Gamma_6,\quad \delta_6=b-\Gamma_6,\quad \delta_7=b\Gamma_7-a_1\Gamma_6-\Gamma_6\Gamma_5,$

$\delta_8=a_1\Gamma_5\Gamma_6,\quad Es_1=\dfrac{Es}{\Gamma_3k^2}, Gr_1=\dfrac{Gr}{\Gamma_3},\quad \Psi_1=\dfrac{Gm}{\Gamma_3},\quad Df_1=\dfrac{Df\,b}{\Psi^*},\quad Df_2=\dfrac{Df_1}{b_1},\quad Df_3=1-Df_2,$

$b_0=Gr_1Df_3,\quad b=Sca_0,\quad b_1=b-a_2,$

In more simple form equations (21-23) are:

$$\begin{aligned}\bar{u}(y,q) &= \Re_1 \cos wt\bar{\Phi}\left(y\sqrt{\Gamma_6},q,0,\Gamma_5,\Gamma_7\right)+\Re_2\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_8,\Gamma_5,\Gamma_7\right)+\Re_3\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_9,\Gamma_5,\Gamma_7\right)\\
&+\Re_4\bar{\Phi}\left(y\sqrt{\Gamma_6},q,0,\Gamma_5,\Gamma_7\right)+\Re_5\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_{16},\Gamma_5,\Gamma_7\right)+\Re_6\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_{17},\Gamma_5,\Gamma_7\right)\\
&+\Re_7\bar{\Phi}\left(y\sqrt{\Gamma_6},q,0,\Gamma_5,\Gamma_7\right)+\Re_8\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_{24},\Gamma_5,\Gamma_7\right)+\Re_9\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_{25},\Gamma_5,\Gamma_7\right)\\
&-\Re_{10}\bar{\Phi}\left(y\sqrt{\Gamma_6},q,0,\Gamma_5,\Gamma_7\right)-\Re_{11}\bar{\Phi}\left(y\sqrt{\Gamma_6},q,\gamma_8,\Gamma_5,\Gamma_7\right)-\Re_1\overline{\Phi_1}\left(y,q,0,b,0,a_1\right)\\
&-\Re_2\overline{\Phi_1}\left(y,q,\gamma_8,b,0,a_1\right)-\Re_3\overline{\Phi_1}\left(y,q,\gamma_9,b,0,a_1\right)-\gamma_3 R_{(\alpha,0)}(-0,t)e^{-ky}\\
&-\gamma_4 R_{(\alpha,0)}(-\gamma_2,t)e^{-ky}-\Re_4\overline{\Phi_1}\left(y,q,0,a_2,0,a_1\right)-\Re_5\overline{\Phi_1}\left(y,q,\gamma_{16},a_2,0,a_1\right)\\
&-\Re_6\overline{\Phi_1}\left(y,q,\gamma_{17},a_2,0,a_1\right)-\Re_7\overline{\Phi_1}\left(y,q,0,b,0,a_1\right)-\Re_8\overline{\Phi_1}\left(y,q,\gamma_{24},b,0,a_1\right)\\
&-\Re_9\overline{\Phi_1}\left(y,q,\gamma_{25},b,0,a_1\right)\end{aligned} \tag{24}$$

$$\bar{\theta}(y,q)=Dr_3\overline{\Phi_1}\left(y,q,0,a_2,0,a_1\right)+Dr_2\overline{\Phi_1}\left(y,q,0,b,0,a_1\right), \tag{25}$$

$$\bar{\phi}(y,q)=\overline{\Phi_1}\left(y,q,0,b,0,a_1\right). \tag{26}$$

Here

$$\bar{\Phi}\left(y,q,s_1,s_2,s_3\right)=\frac{1}{q+s_1}\exp\left(-y\left(\sqrt{\frac{q+s_2}{q+s_4}}\right)\right)$$

$$\overline{\Phi_1}\left(y,q,s_1,s_2,s_3,s_4\right)=\frac{1}{q+s_1}\exp\left(-y\left(\sqrt{\frac{qs_2+s_3}{q+s_4}}\right)\right)$$

Now taking the inverse Laplace transform of the equations (24-26), we get:

$$\begin{aligned}u(y,t) &= \Re_1 \cos wt\Phi*\left(y\sqrt{\Gamma_6},t,0,\Gamma_5,\Gamma_7\right)+\Re_2\Phi\left(y\sqrt{\Gamma_6},t,\gamma_8,\Gamma_5,\Gamma_7\right)+\Re_3\Phi\left(y\sqrt{\Gamma_6},t,\gamma_9,\Gamma_5,\Gamma_7\right)\\
&+\Re_4\Phi\left(y\sqrt{\Gamma_6},t,0,\Gamma_5,\Gamma_7\right)+\Re_5\Phi\left(y\sqrt{\Gamma_6},t,\gamma_{16},\Gamma_5,\Gamma_7\right)+\Re_6\Phi\left(y\sqrt{\Gamma_6},t,\gamma_{17},\Gamma_5,\Gamma_7\right)\\
&+\Re_7\Phi\left(y\sqrt{\Gamma_6},t,0,\Gamma_5,\Gamma_7\right)+\Re_8\Phi\left(y\sqrt{\Gamma_6},t,\gamma_{24},\Gamma_5,\Gamma_7\right)+\Re_9\Phi\left(y\sqrt{\Gamma_6},t,\gamma_{25},\Gamma_5,\Gamma_7\right)\\
&-\Re_{10}\Phi\left(y\sqrt{\Gamma_6},t,0,\Gamma_5,\Gamma_7\right)-\Re_{11}\Phi\left(y\sqrt{\Gamma_6},t,\gamma_8,\Gamma_5,\Gamma_7\right)-\Re_1\Phi_1\left(y,t,0,b,0,a_1\right)\\
&-\Re_2\Phi_1\left(y,t,\gamma_8,b,0,a_1\right)-\Re_3\Phi_1\left(y,t,\gamma_9,b,0,a_1\right)-\gamma_3 R_{(\alpha,0)}(-0,t)e^{-ky}\\
&-\gamma_4 R_{(\alpha,0)}(-\gamma_2,t)e^{-ky}-\Re_4\Phi_1\left(y,t,0,a_2,0,a_1\right)-\Re_5\Phi_1\left(y,t,\gamma_{16},a_2,0,a_1\right)\\
&-\Re_6\Phi_1\left(y,t,\gamma_{17},a_2,0,a_1\right)-\Re_7\Phi_1\left(y,t,0,b,0,a_1\right)-\Re_8\Phi_1\left(y,t,\gamma_{24},b,0,a_1\right)\\
&-\Re_9\Phi_1\left(y,t,\gamma_{25},b,0,a_1\right)\end{aligned} \tag{27}$$

$$\theta(y,t)=Df_3\Phi_1\left(y,t,0,a_2,0,a_1\right)+Df_2\Phi_1\left(y,t,0,b,0,a_1\right), \tag{28}$$

$$\phi\left(y,t\right)=\Phi_1\left(y,t,0,b,0,a_1\right). \tag{29}$$

Where

$$\Phi\left(y,t,s_1.s_2,s_3\right)=\mathrm{e}^{-s_1t-y}-\frac{y\sqrt{s_2-s_3}}{2\sqrt{\pi}}\int_0^\infty\int_0^t\frac{\mathrm{e}^{s_1t}}{\sqrt{t}}\exp\left(s_1t-s_3t-\frac{y^2}{4u}-u\right)I_1\left(2\sqrt{\left(s_2-s_3\right)ut}\right)dtdu.$$

$$\Phi_1\left(z,t,s_1,s_2,s_3,s_4\right)=\mathrm{e}^{-lt-\sqrt{s_2}}-\frac{z\sqrt{s_3-s_2s_4}}{2\sqrt{\pi}}\int_0^t\int_0^\infty\frac{\mathrm{e}^{-s_1t}}{\sqrt{\tau}}\exp\left(s_1\tau-s_4\tau-\frac{z^2}{4u}-s_2u\right)I_1\left(2\sqrt{\left(s_3-s_2s_4\right)u\tau}\right)d\tau du.$$

$$L^{-1}\left(\frac{1}{q^\alpha}\right)=R_{(\alpha,0)}(-0,t),\quad L^{-1}\left(\frac{1}{q^\alpha+\Gamma_9}\right)=R_{(\alpha,0)}(-\Gamma_9,t).$$

Where

$$\gamma_1=1-\Gamma_8,\ \gamma_2=\Gamma_7-\Gamma_8\Gamma_5,\ \gamma_3=\frac{a_1}{a_2},\ \gamma_4=\frac{\gamma_2-\gamma_1a_1}{\gamma_2},\ \gamma_5=\frac{\delta_5}{\delta_4},\ \gamma_6=-\frac{\delta_3}{\delta_4},\ \gamma_7=\frac{\delta_5}{2},$$

$$\gamma_8=\gamma_7+\frac{\sqrt{\gamma_7^2-4\gamma_6}}{2},\ \gamma_9=\gamma_7-\frac{\sqrt{\gamma_7^2-4\gamma_6}}{2},\ \psi_2=\frac{Gm_1}{\delta_4},\ \gamma_{10}=\frac{a_1^2}{\gamma_8\gamma_9},\ \gamma_{11}=\frac{\left(a_1-\gamma_8\right)^2}{\gamma_8\left(\gamma_8-\gamma_9\right)},$$

$$\gamma_{12}=\frac{\left(a_1-\gamma_9\right)^2}{\gamma_9\left(\gamma_9-\gamma_8\right)},\ \gamma_{13}=\frac{\delta_2}{\delta_1},\ \gamma_{14}=\frac{-\delta_3}{\delta_1},\ \psi_3=\frac{b_0}{\delta_1},\ \gamma_{15}=\frac{\gamma_{13}}{2},\ \gamma_{16}=\gamma_{15}+\frac{\sqrt{\gamma_{15}^2-4\gamma_{14}}}{2},$$

$$\gamma_{17}=\gamma_{15}-\frac{\sqrt{\gamma_{15}^2-4\gamma_{14}}}{2},\quad \gamma_{18}=\frac{a_1^2}{\gamma_{16}\gamma_{17}},\quad \gamma_{19}=\frac{\left(a_1-\gamma_{16}\right)^2}{\gamma_{16}\left(\gamma_{16}-\gamma_{17}\right)},\quad \gamma_{20}=\frac{\left(a_1-\gamma_{17}\right)^2}{\gamma_{17}\left(\gamma_{17}-\gamma_{16}\right)},$$

$$\psi_4=\frac{b_1}{\delta_6},\quad \gamma_{21}=\frac{\delta_7}{\delta_6},\quad \gamma_{22}=\frac{-\delta_8}{\delta_6},\quad \gamma_{23}=\frac{\gamma_{21}}{2},\quad \gamma_{24}=\gamma_{23}+\frac{\sqrt{\gamma_{23}^2-4\gamma_{22}}}{2},$$

$$\gamma_{25}=\gamma_{23}-\frac{\sqrt{\gamma_{23}^2-4\gamma_{22}}}{2},\quad \gamma_{26}=\frac{a_1^2}{\gamma_{24}\gamma_{25}},\quad \gamma_{27}=\frac{\left(a_1-\gamma_{24}\right)^2}{\gamma_{24}\left(\gamma_{24}-\gamma_{25}\right)},\quad \gamma_{28}=\frac{\left(a_1-\gamma_{25}\right)^2}{\gamma_{25}\left(\gamma_{25}-\gamma_{24}\right)},$$

$$\Re_1=\psi_2\gamma_{10},\ \Re_2=\psi_2\gamma_{11},\ \Re_3=\psi_3\gamma_{12},\ \Re_4=\psi_3\gamma_{18},\ \Re_5=\psi_3\gamma_{20},\ \Re_6=\psi_3\gamma_{19},$$

$$\Re_7=\psi_4\gamma_{26},\ \Re_8=\psi_4\gamma_{27},\ \Re_9=\psi_4\gamma_{28},\ \Re_{10}=Es_1\gamma_3,\ \Re_{11}=Es_1\gamma_4.$$

5. Solution of the Fractional Model

This section of the present work elaborates the impact of inserted parameters on velocity, temperature, and concentration profile. The local governing equations have been transformed to the fractional Caputo-Fabrizio fractional model. The exact solution of the fractional model has been obtained via the technique of integral transform. The graphical results have been portrayed in response to inserted parameters.

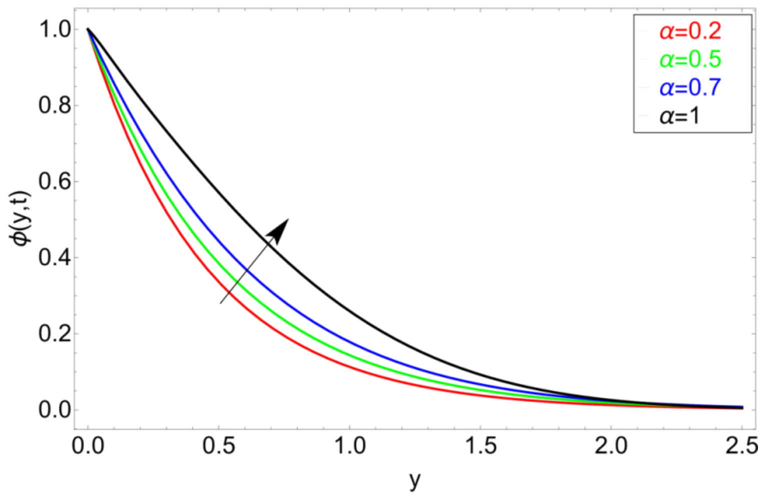

Fig.2. concentration profile in response to fractional parameter α

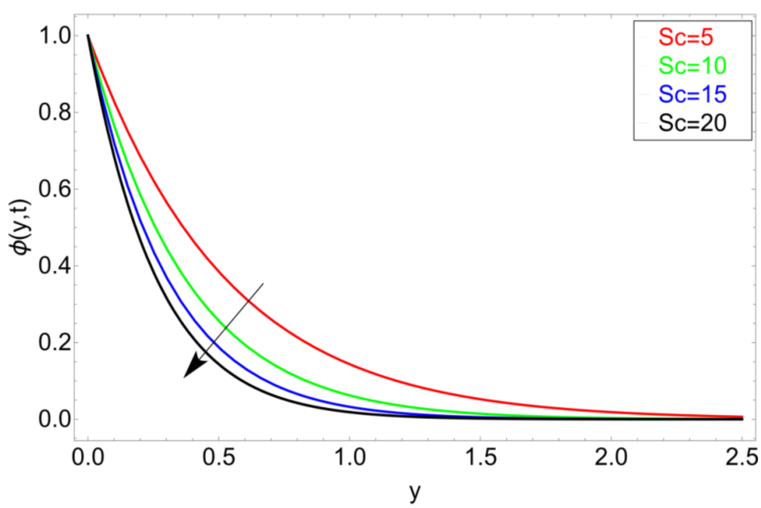

Fig.3. Concentration profile in response of Schmidt number Sc

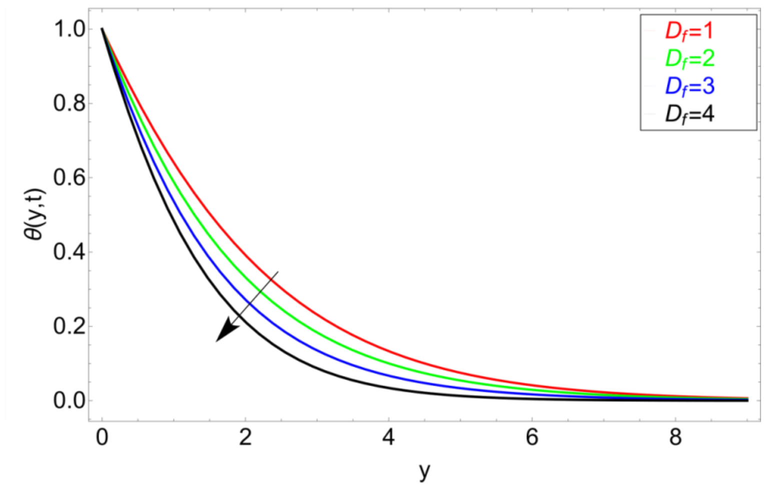

Fig.4. The temperature profile in response to Dufour number *Df*

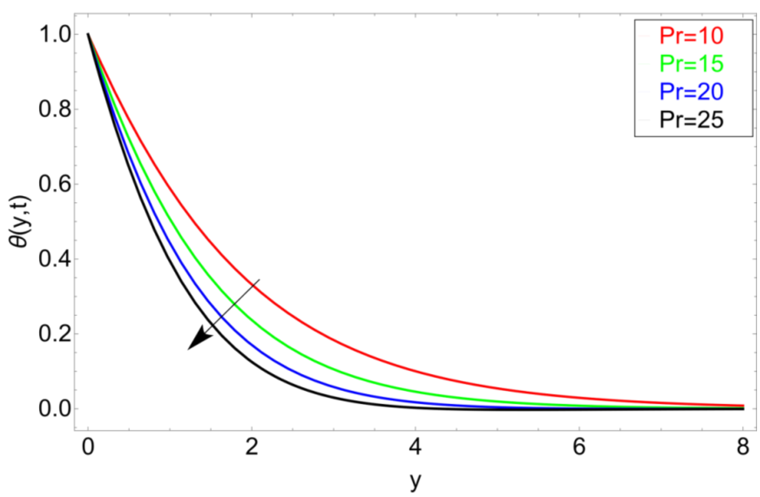

Fig.5. The temperature profile in response to Prandtl number Pr

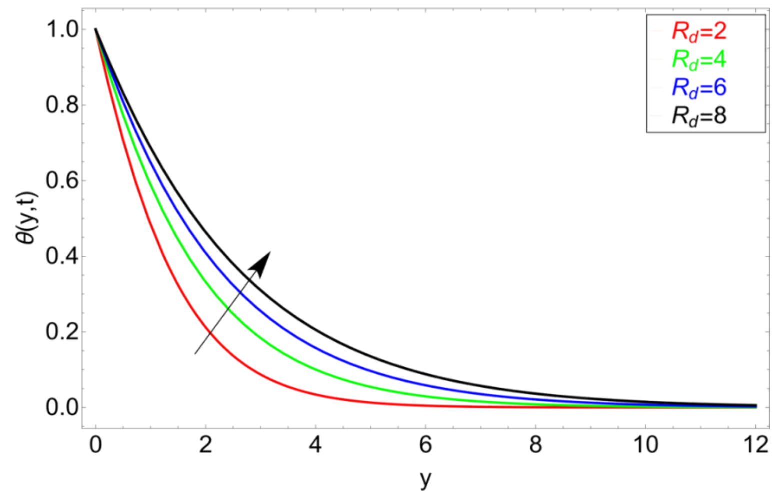

Fig.6. The temperature profile in response to Radiation parameter *Rd*

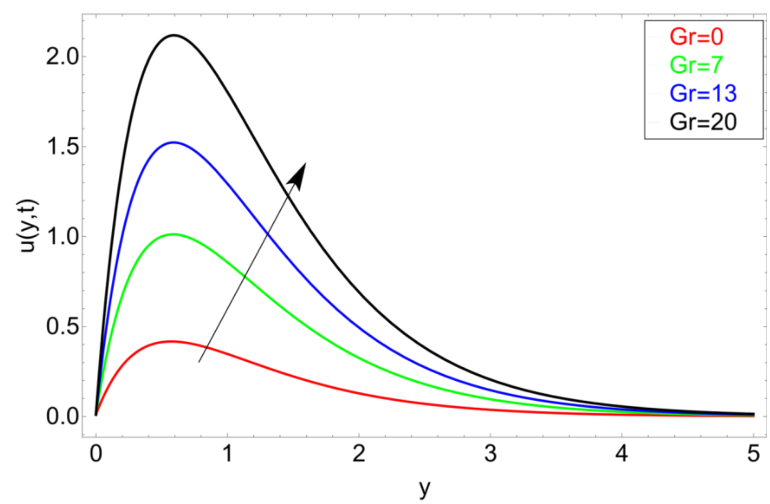

Fig.7. The velocity profile in response of thermal Grashof number *Gr*

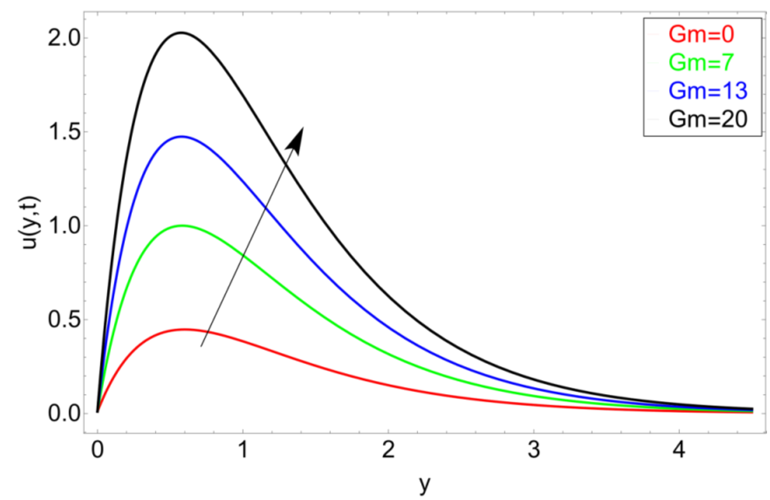

Fig.8. The velocity profile in response of mass Grashof number *Gm*

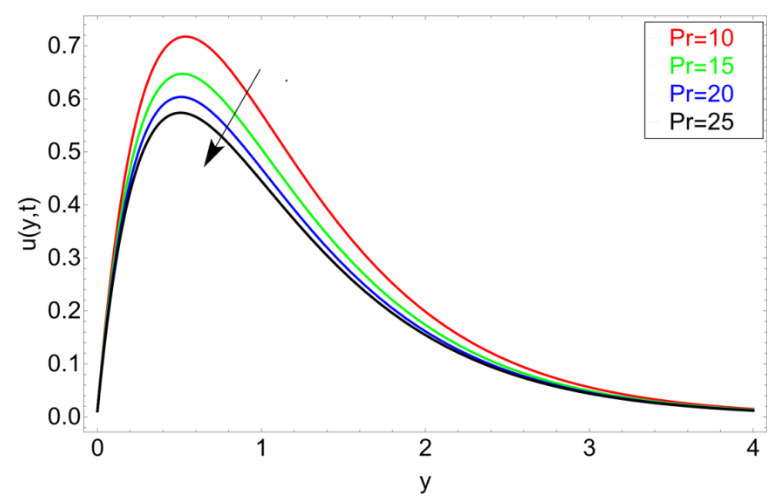

Fig.9. The velocity profile in response to Prandtl number Pr

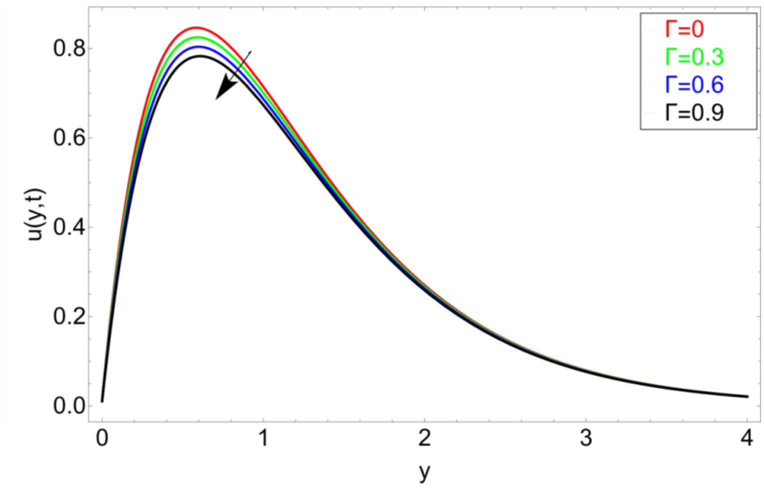

Fig.10. The velocity profile in response to walters'-B parameter Γ

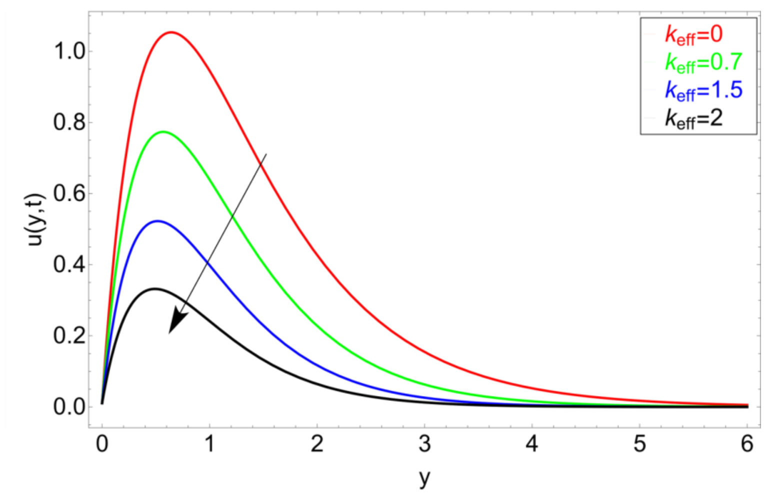

Fig.11. The velocity profile in response of k_{eff}

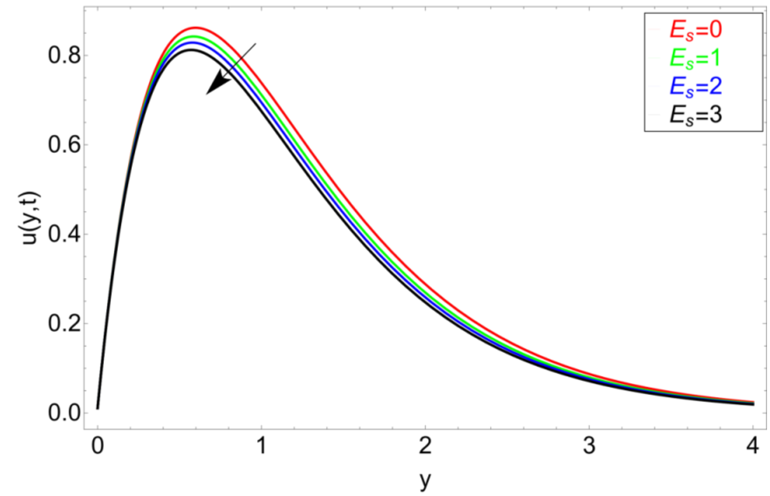

Fig.12. The velocity profile in response to Electro-osmotic parameter *Es*

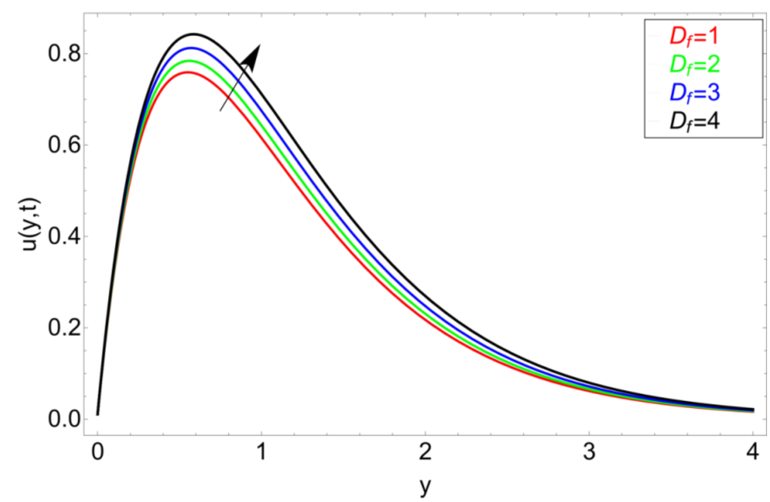

Fig.13. The velocity profile in response of Dufour number *Df*

The behaviour of the concentration profile in response to the fractional parameter α is shown in Fig.2. In non-local derivative, we have only one profile on $\alpha = 1$ for the assumed fluid but in fractional order derivative, we can draw more than one profile on different values of α , and due to this prime advantage, the non-integer order derivative can provide different layers for investigation of the fluid. Experientialists can compare their results with one of the layers that best fits their problem. The behaviour of the concentration profile in response to the Schmidt number has been plotted in Fig.3. it can be noticed from the figure that for increasing magnitude of Sc the profile of concentration decreases and this is because *Sc* has inversely proportional to the mass diffusion rate. Fig.4. shows the effect of Dufour number *Df* on the temperature profile. A fall in the temperature profile can be seen from the figure and it is true because the increasing magnitude of *Df* increases the diffusion rate which consequently affects the temperature of the fluid and hence the temperature of the fluid decrease. Fig.5. illustrate the influence of Pr on the temperature profile. A fall can be noticed from the figure for rising values of Pr and this is due to the reason that thermal forces are weakened for a larger value of Pr. Fig.6 illustrate the effect of radiation parameter Rd on the temperature profile. Temperature profile increases with the increasing values of Rd. Radiation parameter Rd has a direct relationship with the radiation emitting from the fluid and hence temperature profile enhances with increasing values of Rd.

The behaviour of fluid motion in response to thermal and mass Grashof number Gr and Gm have been shown in Fig.7and Fig.8 respectively. Both the graphs depict increasing behaviour for Gr and Gm. The viscous forces in the fluid are reduced as Gr and Gm are increased, causing the fluid to move faster. To show the influence of Prandtl number Pr on the fluid motion, Fig.9 is drawn. Prandtl number represents the ratio between viscous forces and thermal forces. As the magnitude of Pr rises the viscous forces become dominant which leads to a decrease in the velocity profile. Fig.10 is portrayed to check the behaviour of the fluid motion in response to Walters'-B fluid parameter Γ. We can see from the graph that as the value increases, the fluid's speed decreases, which is physically correct because greater values increase viscous forces, causing the fluid to slow down. Fig.11 is drawn to check the impact of k_{eff} on fluid motion. Decreasing velocity profile is observed from these figures for larger values of M. As the magnitude of k_{eff} increases, the resistive forces (Lorentz forces) become strengthen and retard the flow. Fig.12 depicts an increasing behaviour of fluid in response to greater values of Es. The electro-osmotic parameter Es is related to the electric double layer (EDL) when the value Es increases the electric double layer becomes thick which produces more resistive forces in the fluid which leads to slow down the fluid motion. Fig.13 has been plotted to check the behaviour of Df on fluid motion. From the sketch, a rise in the profile of velocity has been reported for larger values of Df because increasing values of Df boost up the rate of mass transfer, and due to this the fluid motion increase.

6. Concluding Remarks

Fractional electro-osmotic flow in the presence of Diffusion-thermo has been inspected on a vertical plate. The classical system of PDEs has been converted to a fractional-order model by the mean of the CF fractional operator. The exact solution has been developed through the Laplace transform technique. The key observations of the present analysis are listed below:

- It has been found that fractional-order explains the memory effect of the considered fluid which is not possible by the classical mathematical model.
- Velocity, temperature, and concentration are increasing functions of the fractional parameter α .
- Increasing values of Rd, Gr, Gm, and Df boost up the temperature and velocity profile.

Increasing values of Sc, Es, Pr, k_{eff}, and Γ declines the profile of temperature, concentration, and velocity profile.

References

[1] F. F. Reuss. (1809). Sur un nouvel effet de llectricit galvanique, Nouveaux de la Socit imperial des naturlistes de Moscou, 2, (1809) 327-337.

[2] Asadi, A., Huat, B. B., Nahazanan, H., & Keykhah, H. A. (2013). Theory of electroosmosis in soil. International Journal of Electrochemical Science, 8(1), 1016-1025.

[3] Gray, D. H. (1970). Electrochemical hardening of clay soils. Geotechnique, 20(1), 81-93.

[4] D.H., Gray, and H.K. (1967), Mitchell, Journal of the Soil Mechanics and Foundations, ASCE 93 (1967) 209-236.

[5] Das, S., & Chakraborty, S. (2006),. Analytical solutions for velocity, temperature and concentration distribution in electroosmotic microchannel flow of a non-Newtonian bio-fluid. Analytica Chimica Acta, 559(1) 15-24.

[6] Zhao, C., Zholkovskij, E., Masliyah, J. H., & Yang, C. (2008), Analysis of electroosmotic flow of power-law fluids in a slit microchannel. Journal of colloid and interface science, 326(2), 503-510.

[7] Tang, G. H., Li, X. F., He, Y. L., & Tao, W. Q. (2009). Electroosmotic flow of non-Newtonian fluid in microchannels. Journal of Non-Newtonian Fluid Mechanics, 157(1-2) 133-137.

[8] Zhao, C., & Yang, C. (2012). Joule heating induced heat transfer for the electroosmotic flow of power-law fluids in a microcapillary. International Journal of Heat and Mass Transfer, 55(7-8), 2044-2051.

[9] Liu, Q. S., Jian, Y. J., & Yang, L. G. (2011). Time periodic electroosmotic flow of the generalized Maxwell fluids between two micro-parallel plates. Journal of Non-Newtonian Fluid Mechanics, 166(9-10), 478-486.

[10] Liu, Q., Jian, Y., & Yang, L. (2011). The alternating current electroosmotic flow of the Jeffreys fluids through a slit microchannel. Physics of Fluids, 23(10), 01-13.

[11] Ali, F., Iftikhar, M., Khan, I., Sheikh, N. A., & Nisar, K. S. (2020). Time fractional analysis of electro-osmotic flow of Walters' sB fluid with time-dependent temperature and concentration. Alexandria Engineering Journal, 59(1), 25-38.

[12] Murtaza, S., Iftekhar, M., Ali, F., & Khan, I. (2020). Exact Analysis of Non-Linear Electro-Osmotic Flow of Generalized Maxwell Nanofluid: Applications in Concrete Based Nano-Materials. IEEE Access, 8, 96738-96747.

[13] Machado, J. T., Galhano, A. M., & Trujillo, J. J. (2013). Science metrics on fractional calculus development since 1966. Fractional Calculus and Applied Analysis, 16(2), 479-500.

[14] Oldham, K., & Spanier, J. (1974). The fractional calculus theory and applications of differentiation and integration to arbitrary order (Vol. 111). Elsevier.

[15] Murtaza, S., Farhad Ali, A., Sheikh, N. A., Khan, I., & Nisar, K. S. (2020). Exact Analysis of Non-Linear Fractionalized Jeffrey Fluid. A Novel Approach of Atangana-Baleanu Fractional Model. CMC-COMPUTERS MATERIALS & CONTINUA, 65(3), 2033-2047.

[16] Murtaza, S., Ali, F., Sheikh, N. A., Khan, I., & Nisar, K. S. (2021). Analysis of Silver Nanoparticles in Engine Oil: Atangana-Baleanu Fractional Model. CMC-COMPUTERS MATERIALS & CONTINUA, 67(3), 2915-2932.

[17] Ahmad, J., Ali, F., Murtaza, S., & Khan, I. (2021). Caputo Time Fractional Model Based on Generalized Fourier's and Fick's Laws for Jeffrey Nanofluid: Applications in Automobiles. Mathematical Problems in Engineering, 2021.

[18] Caputo, M., & Fabrizio, (2015). M. A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl, 1(2), 1-13.

[19] Zafar, A. A., & Fetecau, C. (2016). Flow over an infinite plate of a viscous fluid with a non-integer order derivative without singular kernel. Alexandria Engineering Journal, 55(3), 2789-2796.

[20] Gambo, Y. Y., Jarad, F., Baleanu, D., & Abdeljawad, T. (2014). On Caputo modification of the Hadamard fractional derivatives. Advances in Difference Equations, 2014(1), 10-21.

[21] Ali, F., Saqib, M., Khan, I., & Sheikh, N. A. (2016). Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters'-B fluid model. The European Physical Journal Plus, 131(10), 377.

[22] Abdeljawad, T., & Baleanu, D. (2017). On fractional derivatives with exponential kernel and their discrete versions. Reports on Mathematical Physics, 80(1), 11-27.

[23] Sheikh, N. A., Ali, F., Khan, I., & Saqib, M. (2016). A modern approach of Caputo–Fabrizio time-fractional derivative to MHD free convection flow of generalized second-grade fluid in a porous medium. Neural Computing and Applications, 1-11.

[24] Ali, F., Ali, F., Sheikh, N. A., Khan, I., & Nisar, K. S. (2020). Caputo–Fabrizio fractional derivatives modeling of transient MHD Brinkman nanoliquid: Applications in food technology. Chaos, Solitons & Fractals, 131, 109489.

[25] Ali, F., Khan, I., Sheikh, N. A., Gohar, M., & Tlili, I. (2018). Effects of different shaped nanoparticles on the performance of engine-oil and kerosene-oil: a generalized Brinkman-type fluid model with non-singular kernel. Scientific reports, 8(1), 1-13.

[26] Ali, F., Murtaza, S., Sheikh, N. A., & Khan, I. (2019). Heat transfer analysis of generalized Jeffery nanofluid in a rotating frame: Atangana–Balaenu and Caputo–Fabrizio fractional models. Chaos, Solitons & Fractals, 129, 1-15.

[27] Saqib, M., Ali, F., Khan, I., Sheikh, N. A., & Jan, S. A. A. (2018). Exact solutions for free convection flow of generalized Jeffrey fluid: a Caputo-Fabrizio fractional model. Alexandria engineering journal, 57(3), 1849-1858.

[28] Ali, F., Saqib, M., Khan, I., & Sheikh, N. A. (2016). Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters'-B fluid model. The European Physical Journal Plus, 131(10), 377.

[29] Sheikh, N. A., Ali, F., Khan, I., & Saqib, M. (2018). A modern approach of Caputo–Fabrizio time-fractional derivative to MHD free convection flow of generalized second-grade fluid in a porous medium. Neural Computing and Applications, 30(6), 1865-1875.

[30] Turkyilmazoglu, M., & Pop, I. (2013). Heat and mass transfer of unsteady natural convection flow of some nanofluids past a vertical infinite flat plate with radiation effect. International Journal of Heat and Mass Transfer, 59, 167-171.