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HYBRID-BLOCK NUMERICAL METHOD FOR SOLVING SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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A B S T R A C T

In this work, a new hybrid block method for solving second order initial value problems of ordinary differential equations is developed. The derivation is achieved via multistep collocation approach with the use of approximated power series as the basis function. The discrete schemes and its derivatives are derived by evaluating the basis function at the grid and non-grid points which are used to form the block. In other to examine the efficiency of the new developed block method, it is applied to second order initial value problems and the results generated revealed the accuracy of the method over the existing methods.
In this work, a new hybrid-block method for the solution of second order ordinary differential equation is developed using power series as the basis function. The developed scheme was used to solve some problems and the result compare with existing results to ascertain the superiority of the new method.

1. Introduction

In order to numerically and more accurately solve ordinary differential equations arising from science, social sciences and engineering, which most times do not have analytical solutions, many scholars had proposed several different numerical methods such as linear multistep, Euler, Runge-Kutta, hybrid and block methods depending on the nature and type of the differential equation to be solved. This work is focused on the numerical solution of second order differential equations.

$$\left.\begin{array}{c} y'' = f(x,y,y'),\ \ y(a) = \mu_0,\ \ y(a) = \mu_1 \\ x \in [a,b] \end{array}\right\} \qquad (1)$$

Though, these methods have their drawbacks but they can be circumvented or improve on as demonstrated in this work.

2. Literature Review

Many Scholars such ([6], [15,16], [9]) have suggested in the literature that a better alternative is to solve equation (1) directly without first reducing it to a system of first order ordinary differential equations.

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In [3], Awoyemi adopted the method and proposed a two-step hybrid multistep method with continuous coefficients for the solution of (1) based on collocation at selected grid points and using off-grid points to upgrade the order of the method and to provide one additional interpolation point and implemented on the hybrid predictor-corrector mode. Later, [10], hybrid method of order four was used to generate starting values for Numerov method. Many scholars which include but not limited to ([5],[8],[17]) had studied hybrid methods.

Fatunla [1,2] block method for special second order differential equations which was later developed by [4,7,11] was proposed. Parallel block methods in explicit and implicit for the solution of higher order differential equations where suitable interpolating polynomial was used to approximate the derivative function with a specified interval of integration was obtained. Many other scholars such as [12, 18], [13], [19] have adopted block methods where the derivative function was interpolated using Lagrange interpolation. These methods have largely focused on solving only special type ordinary differential equations with very few attempts in favour of (1). [14] and [20] have proposed five-step and four-step self-starting methods which adopt continuous linear multistep method to obtain finite difference methods applied respectively as a block for the direct solution of (1). Recently, ([21], [22], [23]) adopted hybrid-block method for the solution of second order ordinary differential equations and the results were found to be accurate and the scheme efficient compare to block methods.

While all the methods have their qualities and are very robust, we propose in this research, a new hybrid-block method that harness the qualities of the existing methods for the direct solution of (1).

3. Methodology

In this section, a new Hybrid-Block scheme for the solution of second order ordinary differential equation is developed. The idea is to collocate the assumed function $y(x)$ at three points and interpolate at two points, which leaves us with five equations. Thus, evaluating at these points to derive the method.

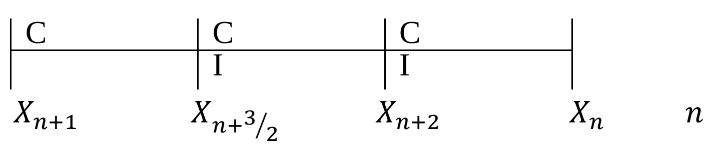

Fig 1: collocation points

3.1. Derivation of the Hybrid-Block Method.

Using power series as basis function we have

$$y(x) = \sum_{i=0}^{4} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{3.1}$$

Then

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \tag{3.2}$$

$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 \tag{3.3}$$

Collocating equation (3.3) at $x = x_n,\ x = x_{n+1}$ and $x = x_{n+2}$ we have

$$y''(x_n) = 2a_2 + 6a_3x_n + 12a_4x_n^2 \tag{3.4}$$

$$y''(x_{n+1}) = 2a_2 + 6a_3x_{n+1} + 12a_4x_{n+1}^2 \tag{3.5}$$

$$y''(x_{n+2}) = 2a_2 + 6a_3x_{n+2} + 12a_4x_{n+2}^2 \tag{3.6}$$

Also interpolating (3.1) at $x = x_{n+1}$ and at $x = x_{n+3/2}$ we have

$$y(x_{n+1}) = a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_{n+1}^4 \tag{3.7}$$

$$y\left(x_{n+3/2}\right) = a_0 + a_1x_{n+3/2} + a_2x_{n+3/2}^2 + a_3x_{n+3/2}^3 + a_4x_{n+3/2}^4 \tag{3.8}$$

The resulting matrix (of the form$Ax = b$) from the system of equations (3.4), (3.5), (3.6), (3.7) and (3.8) is

$$\begin{bmatrix} 0 & 0 & 2 & 6x_n & 12x_n^2 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+3/2} & x_{n+3/2}^2 & x_{n+3/2}^3 & x_{n+3/2}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y''(x_n) \\ y''(x_{n+1}) \\ y''(x_{n+2}) \\ y(x_{n+1}) \\ y\left(x_{n+3/2}\right) \end{bmatrix} \tag{3.9}$$

Solving (3.9) using MatLab to find $a_i,\ i = 0,1,2,3,4$ we have

$$a_0 = \frac{[(21h^4 + 77h^3x_n + 96h^2x_n^2 + 48hx_n^3 + 8x_n^4)f_n]}{192h^2} + \frac{[(-3h^4 - 11h^3x_n + 16hx_n^3 + 8x_n^4)f_{n+2}]}{192h^2}$$
$$- \frac{[(-63h^4 - 87h^3x_n + 32hx_n^3 + 8x_n^4)f_{n+1}]}{96h^2} + \frac{[(3h + 2x_n)y_{n+1}]}{h} - \frac{[2(h + x_n)y_{n+3/2}]}{h}$$
$$= \frac{(21h^4 + 77h^3x_n + 96h^2x_n^2 + 48hx_n^3 + 8x_n^4)f_n + (-3h^4 - 11h^3x_n + 16hx_n^3 + 8x_n^4)f_{n+2}}{192h^2}$$
$$- \frac{[(-63h^4 - 87h^3x_n + 32hx_n^3 + 8x_n^4)f_{n+1}]}{96h^2} + \frac{(3h + 2x_n)y_{n+1} - \left[2(h + x_n)y_{n+3/2}\right]}{h}$$
$$= \frac{21h^4f_n + 77h^3x_nf_n + 96h^2x_n^2f_n + 48hx_n^3f_n + 8x_n^4f_n - 3h^4f_{n+2} - 11h^3x_nf_{n+2} + 16hx_n^3f_{n+2} + 8x_n^4f_{n+2}}{192h^2}$$
$$+ \frac{63h^4f_{n+1} + 87h^3x_nf_{n+1} - 32hx_n^3f_{n+1} - 8x_n^4f_{n+1}}{96h^2} + \frac{3hy_{n+1} + 2x_ny_{n+1} - 2hy_{n+3/2} - 2x_ny_{n+3/2}}{h}$$

$$a_0 = \frac{h^4(21f_n - 3f_{n+2}) + h^3x_n(77f_n - 11f_{n+2}) + 96h^2x_n^2f_n + x_n^3(48hf_n + f_{n+2}16h) + x_n^4(8f_n + 8f_{n+2})}{192h^2}$$
$$+ \frac{63h^4f_{n+1} + 87h^3x_nf_{n+1} - 32hx_n^3f_{n+1} - 8x_n^4f_{n+1}}{96h^2} + \frac{h\left(3y_{n+1} - 2y_{n+3/2}\right) + x_n\left(y_{n+1} - 2y_{n+3/2}\right)}{h}$$
$$a_1 = \frac{2y_{n+3/2}}{h} - \frac{2y_{n+1}}{h} - \frac{[(77h^3 + 192h^2x_n + 144hx_n^2 + 32x_n^3)f_n]}{192h^2} - \frac{[(-11h^3 + 48hx_n^2 + 32x_n^3)f_{n+2}]}{192h^2}$$
$$+ \frac{[(-87h^3 + 96hx_n^2 + 32x_n^3)f_{n+1}]}{96h^2}$$

$$= \frac{2\left(y_{n+3/2} - y_{n+1}\right)}{h} - \frac{77h^3 f_n + 192h^2 x_n f_n + 144hx_n^2 f_n + 32x_n^3 f_n}{192h^2} + \frac{11h^3 f_{n+2} - 48hx_n^2 f_{n+2} - 32x_n^3 f_{n+2}}{192h^2} - \frac{87h^3 f_{n+1} + 96hx_n^2 f_{n+1} + 32x_n^3 f_{n+1}}{96h^2}$$

$$a_1 = \frac{2\left(y_{n+3/2} - y_{n+1}\right)}{h} - \frac{h^3(77f_n - 11f_{n+2}) + 192h^2 x_n f_n + hx_n^2(144f_n - 48f_{n+2}) + 32x_n^3(f_n - f_{n+2})}{192h^2} - \frac{87h^3 f_{n+1} + 96hx_n^2 f_{n+1} + 32x_n^3 f_{n+1}}{96h^2}$$

$$a_2 = \frac{(x_n^2 + hx_n)f_{n+2}}{4h^2} - \frac{[(x_n^2 + 2hx_n)f_{n+1}]}{2h^2} + \frac{(2h^2 + 3hx_n + x_n^2 f_n)}{4h^2}$$

$$= \frac{x_n^2 f_{n+2} + hx_n f_{n+2} + 2h^2 f_n + 3hx_n f_n + x_n^2 f_n}{4h^2} - \frac{x_n^2 f_{n+1} - 2hx_n f_{n+1}}{2h^2}$$

$$a_2 = \frac{x_n^2(f_{n+2} + f_n) + hx_n(f_{n+2} + f_n) + f_n 2h^2}{4h^2} - \frac{x_n^2 f_{n+1} - 2hx_n f_{n+1}}{2h^2}$$

$$a_3 = \frac{(h + x_n)f_{n+1}}{3h^2} - \frac{[(3h + 2x_n)f_n]}{12h^2} - \frac{[(h + 2x_n)f_{n+2}]}{12h^2}$$

$$= \frac{(h + x_n)f_{n+1}}{3h^2} - \frac{3hf_n - 2x_n f_n - hf_{n+2} + 2x_n f_{n+2}}{12h^2}$$

$$a_3 = \frac{(h + x_n)f_{n+1}}{3h^2} - \frac{h(3f_n - f_{n+2}) - 2x_n(f_n + f_{n+2})}{12h^2}$$

$$a_4 = \frac{f_n}{24h^2} - \frac{f_{n+1}}{12h^2} + \frac{f_{n+2}}{24h^2}$$

$$a_4 = \frac{f_n + f_{n+2}}{24h^2} - \frac{f_{n+1}}{12h^2}$$

Substituting $a_0, a_1, a_2, a_3 \; and \; a_4$ into (3.1)

$$y(x) = \frac{h^4(21f_n - 3f_{n+2}) + h^3 x_n(77f_n - 11f_{n+2}) + 96h^2 x_n^2 f_n + x_n^3(48hf_n + 16hf_{n+2}) + x_n^4(8f_n + 8f_{n+2})}{192h^2} + \frac{63h^4 f_{n+1} + 87h^3 x_n f_{n+1} - 32hx_n^3 f_{n+1} - 8x_n^4 f_{n+1}}{96h^2} + \frac{h\left(3y_{n+1} - 2y_{n+3/2}\right) + x_n\left(y_{n+1} - 2y_{n+3/2}\right)}{h} + \frac{2x\left(y_{n+3/2} - y_{n+1}\right)}{h} - \frac{xh^3(77f_n - 11f_{n+2}) + 192h^2 x_n x f_n + xhx_n^2(144f_n - 48f_{n+2}) + 32x_n^3 x(f_n - f_{n+2})}{192h^2} - \frac{87h^3 x f_{n+1} + 96hx_n^2 x f_{n+1} + 32x_n^3 x f_{n+1}}{96h^2}$$

$$+\frac{x^2x_n^2(f_{n+2}+f_n)+x^2hx_n(f_{n+2}+f_n)+2x^2f_nh^2}{4\text{h}^2}-\frac{x^2x_n^2f_{n+1}-2hx_nx^2f_{n+1}}{2\text{h}^2}+\frac{x^3(h+x_n)f_{n+1}}{3\text{h}^2}$$
$$-\frac{x^3h(3f_n-f_{n+2})-2x_nx^3(f_n+f_{n+2})}{12\text{h}^2}+\frac{x^4f_n+x^4f_{n+2}}{24\text{h}^2}$$
$$-\frac{x^4f_{n+1}}{12\text{h}^2} \qquad (3.10)$$

let

$$t=\frac{x-(x_{n+k-1})}{h}$$

Then,

$$t=\frac{x-(x_{n+2-1})}{h}, \qquad when\ k=2$$

$$t=\frac{x-(x_{n+1})}{h}=\frac{x-(x_n+h)}{h}, \qquad since\ x_{n+1}=x_n+h$$

$$th=\ x-x_n-h$$

$$x=th+x_n+h$$

$$x=x_n, \qquad when\ t=-1$$

$$x=x_{n+2}=x_n+2h, \qquad when\ t=1$$

From (3.10), the continuous scheme becomes

$$y(x_n)=y_n=y_{n+1}-2ty_{n+1}+2ty_{n+3/2}+\frac{(th^2f_n)}{64}-\frac{(23th^2f_{n+1})}{96}-\frac{(5th^2f_{n+2})}{192}-\frac{(t^3h^2f_n)}{12}+\frac{(t^2h^2f_{n+1})}{2}+\frac{(t^4h^2f_n)}{24}$$
$$-\frac{(t^4h^2f_{n+1})}{12}+\frac{(t^3h^2f_{n+2})}{12}$$
$$+\frac{(t^4h^2f_{n+2})}{24} \qquad (3.11)$$

The derivative of (3.11) with respect to t becomes

$$y'(x_n)=y_n'=\frac{h(3f_n-46f_{n+1}-5f_{n+2}+192tf_{n+1}-48t^2f_n+32t^3f_n-64t^3f_{n+1}+48t^2f_{n+2}+32t^3f_{n+2}}{192}$$
$$-\frac{2y_{n+1}-2y_{n+3/2}}{h} \qquad (3.12)$$

The discrete scheme when $t=-1$

$$y(x_n)=y_n=3y_{n+1}-2y_{n+3/2}+\frac{h^2}{64}[7f_n+42f_{n+1}-f_{n+2}] \qquad (3.13)$$

Also when $t=1$ we have

$$y_{n+2}=2y_{n+3/2}-y_{n+1}-\frac{h^2}{192}[5f_n-34f_{n+1}-19f_{n+2}] \qquad (3.14)$$

Evaluating the derivative of the discrete scheme at $x=x_n$ and $x=x_{n+2}$

At $x=x_n$

$$y_n'=\frac{-(2y_{n+1}-2y_{n+3/2})}{h}-\frac{(h(77f_n+174f_{n+1}-11f_{n+2}))}{192}$$

$$hy_n'=2y_{n+3/2}-2y_{n+1}-\frac{h^2}{192}[77f_n+174f_{n+1}-11f_{n+2}] \qquad (3.15)$$

At $x=x_{n+2}$

$$y_{n+2}'=\frac{-(2y_{n+1}-2y_{n+3/2})}{h}+\frac{(h(82f_{n+1}-13f_n+75f_{n+2}))}{192}$$

$$hy'_{n+2} = 2y_{n+3/2} - 2y_{n+1} + \frac{h^2}{192}[-13f_n + 82f_{n+1} + 72f_{n+2}] \qquad (3.16)$$

Expressing (3.13), (3.14) and (3.15) in matrix form

$$\begin{bmatrix} -3 & 2 & 0 \\ 1 & -2 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n-3/2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y'_{n-3/2} \\ y'_{n-1} \\ y'_n \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 7/64 \\ 0 & 0 & -5/192 \\ 0 & 0 & -77/192 \end{bmatrix} \begin{bmatrix} f_{n-3/2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} 42/64 & 0 & -1/64 \\ 34/192 & 0 & 19/192 \\ -174/192 & 0 & 11/192 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{bmatrix} \qquad (3.17)$$

Let $A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -2 & 1 \\ 2 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$

$D = \begin{bmatrix} 0 & 0 & 7/64 \\ 0 & 0 & -5/192 \\ 0 & 0 & -77/192 \end{bmatrix}, E = \begin{bmatrix} 42/64 & 0 & -1/64 \\ 34/192 & 0 & 19/192 \\ -174/192 & 0 & 11/192 \end{bmatrix}$

Then, $A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -3/2 \\ -1 & 1 & -2 \end{bmatrix}$

Multiply through (3.19) by the inverse of A we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3/2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3/2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y'_{n-3/2} \\ y'_{n-1} \\ y'_n \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 7/24 \\ 0 & 0 & 63/128 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} f_{n-3/2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} 1/4 & 0 & -1/24 \\ 45/64 & 0 & -9/128 \\ 4/3 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \end{bmatrix}$$

This implies that

$$y_{n+1} = y_n + hy'_n + h^2\left[\frac{7}{24}f_n + \frac{1}{4}f_{n+1} - \frac{1}{24}f_{n+2}\right]$$

$$y_{n+3/2} = y_n + \frac{3}{2}hy'_n + h^2\left[\frac{63}{128}f_n + \frac{45}{64}f_{n+1} - \frac{9}{128}f_{n+2}\right]$$

$$y_{n+2} = y_n + 2hy'_n + h^2\left[\frac{2}{3}f_n + \frac{4}{3}f_{n+2}\right]$$

This also implies

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{24}[7f_n + 6f_{n+1} - f_{n+2}]$$

$$y_{n+3/2} = y_n + \frac{3}{2}hy'_n + \frac{9h^2}{128}[7f_n + 10f_{n+1} - f_{n+2}]$$

$$y_{n+2} = y_n + 2hy'_n + \frac{h^2}{3}[2f_n + 4f_{n+2}]$$

The Derivatives,

$$y'_{n+1} = y'_n + \frac{h}{12}[5f_n + 8f_{n+1} - f_{n+2}]$$

$$y'_{n+3/2} = y'_n + \frac{h}{8}[3f_n + 9f_{n+1}]$$

$$y'_{n+2} = y'_n + \frac{h}{3}[f_n + 4f_{n+1} + f_{n+2}]$$

This is the derived Hybrid-Block Method

4. Numerical Examples

In order to ascertain the efficiency of this method, numerical experiment of some problems are performed and the results compared with that of the earlier literature.

4.1. Example 1

$$y'' - x(y')^2 = 0, y(0), y'(0) = \frac{1}{2}, h = \frac{1}{30},$$

Exact solution;

$$y(x) = 1 + \frac{1}{2}In\left(\frac{2+x}{2-x}\right)$$

Table 1: Comparison of the new method with Badmus and Yahaya. (2009) for solving Problem two.

x	**Theoretical Solution**	**Computed Solution**	**Error**	
			New method	**Badmus and Yahaya (2009), $k = 5$**
0.1	1.050041729278491400	1.050048113815469600	$6.38 * 10^{-06}$	$5.89 * 10^{-06}$
0.2	1.100335347731075600	1.100354277282312500	$1.89 * 10^{-05}$	$8.24 * 10^{-05}$
0.3	1.151140435936466800	1.151166322749548800	$2.59 * 10^{-05}$	$3.46 * 10^{-04}$
0.4	1.202732554054082100	1.202772918188963500	$4.04 * 10^{-05}$	$7.52 * 10^{-04}$
0.5	1.255412811882995200	1.255461682142221100	$4.89 * 10^{-05}$	$1.38 * 10^{-03}$

4.2. Example 2

$$y'' - 100y = 0, y(0) = 1, y'(0) = -10, \qquad h = 0.01$$

Exact Solution: $y(x) = e^{-10x}$

Table 2: Comparison of the new method with Awari et al. (2004) for solving Problem one

x	Theoretical Solution	Computed Solution	Error in new method	Error in Awari et al (2004) $k = 4$
0.0100000	0.904837418035959520	0.904837622396990730	$2.04 * 10^{-07}$	$1.11 * 10^{-05}$
0.0200000	0.818730753077981820	0.818730938148340860	$1.85 * 10^{-07}$	$3.14 * 10^{-05}$
0.0300000	0.740818220681717880	0.740818388435980580	$1.68 * 10^{-07}$	$5.27 * 10^{-05}$
0.0400000	0.670320046035639330	0.670320198263434810	$1.52 * 10^{-07}$	$7.45 * 10^{-05}$
0.0500000	0.606530659712633420	0.606530798037663300	$1.38 * 10^{-07}$	$8.23 * 10^{-05}$
0.0600000	0.548811636094026390	0.548811761991317980	$1.26 * 10^{-07}$	$9.71 * 10^{-05}$
0.0700000	0.496585303791409470	0.496585418602985000	$1.15 * 10^{-07}$	$1.13 * 10^{-04}$
0.0800000	0.449328964117221560	0.449329069066350170	$1.05 * 10^{-07}$	$1.31 * 10^{-04}$
0.0900000	0.406569659740599050	0.405999744850104430	$5.70 * 10^{-04}$	$1.36 * 10^{-04}$
0.1000000	0.367879441171442330	0.367363762289662830	$5.17 * 10^{-04}$	$1.47 * 10^{-04}$

5. Conclusion

It is observed from Table 1 that the result obtained from the method is more efficient when compared to that of Badmus and Yahaya (2009). However, even though the error in the Block method proposed by Badmus and Yahaya (2009) seemed to have produced a good result at it points of evaluation, it should be noticed that the method had step number $k = 5$ against our method with step number $k = 2$.
Also, in Table 2, it is also observed that the error in Awari et al (2004) also seemed to have a good result at its points of evaluation. It should also be noticed that the method in Awari et al (2004) had step number $k = 4$ against our method with step number $k = 2$.
All computations were carried out using MATLAB 2008 and executed on Windows8 operating system.

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